Mathematics in Trades and Life

Mathematics in Trades and Life

Mark Fitch
University of Alaska Anchorage

Megan Ossiander-Gobeille
University of Alaska Anchorage

13 August 2025

Edition: Fall 2025

Website: uaa.alaska.edu

©2021 Mark Fitch, Megan Ossiander-Gobeille

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the appendix entitled "GNU Free Documentation License." All trademarks™ are the registered® marks of their respective owners.

Acknowledgements

This text began as class presentation notes from Megan Gobeille. Its development was greatly accelerated with the support of a sabbatical by the University of Alaska Anchorage. The following people have provided valuable input.

- Megan Ossiander-Gobeille
- Deb Crawford

For Students

This book is intended for use as lessons in a course that emphasizes building the skills to read and use mathematics (such as in a technical manual), and to recognize mathematical concepts in things you encounter in life.

This book is written to demonstrate effective thought processes, and provide the reader with practice learning to analyze examples and apply examples to similar—not solely identical—problems.

The general approach is to provide initial examples, present general concepts used in the examples, and then provide a chance to practice the concept or skill. There are two types of special examples. One demonstrates the entire thought process used to figure out what to do. These are much longer and ill suited to use as examples when working on an assignment. Another type of special example illustrates using an example (examples on how to use an example). These are also longer as they show the thought process for connecting pieces of a question to a sufficiently similar problem.

By design there are exercises for which there is no step-by-step example in the section. This book is not written as a how to manual for specific applications. When you encounter these problems, as needed, refer to the examples illustrating thought processes. Mimic the thinking skills. You will go farther in life with general thinking skills than you will with memorizing specific processes.

The topics include

1. Interpret data in various formats and analyze mathematical models
2. Read and use mathematical models in a technical document
3. Communicate results in mathematical notation and in language appropriate to the technical field

You will learn to work with the following mathematical concepts.

1. Precision and accuracy
   (a) Rounding (skill)
   (b) Significant Figures (skill)
   (c) Determining appropriate rounding from context (critical thinking)
2. Proportions
   (a) Set up and solve proportions (skill)
   (b) Calculate Percentages (skill)
   (c) Understand and interpret percentages (critical thinking)
   (d) Unit conversion (skill)
3. Rates
   (a) Identify rates as linear, quadratic, exponential, or other (critical thinking)
   (b) Identify data varrying directly or indirectly (critical thinking)

4. Solving
    (a) Solve linear, rational, quadratic, and exponential equations and formulas (skill)
    (b) Solve a system of linear equations (skill)
5. Models
    (a) Read and interpret models (critical thinking)
    (b) Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
6. Geometric Reasoning
    (a) Identify shapes and apply their properties (skill)
    (b) Analyze right triangles (skill)
    (c) Analyze non-right triangles (skill)
    (d) Identify properties of sine and cosine functions (skill)

For Faculty

This book is designed to be used for classes supporting trade programs in a variety of fields and also to satisfy baccalaureate general education requirements.

Because of the focus on trade programs, choices of mathematical topics and the extent to which they are covered is more limited than in algebra or precalculus texts. Only those topics that are used in the selected trades and some extensions for quantitative literacy are included. Also examples are designed to look like how a person in a trade would actually perform the calculations. This includes the assumption that all arithmetic will be performed by a device and hence almost all numbers will be in decimal notation. For this reason rounding is covered and consistently required.

The inclusion of general education outcomes motivates the "life" portion of the book. These applications are not restricted to use in one or more trades, rather they enable any person to process simple phenomena they experience. The quantitative literacy sections also contribute to the general education goals. Many topics will conclude with activities designed to help readers see limitations of models and learn to process mathematical information with an appropriately critical eye.

This book is also designed to be used in conjunction with active learning pedagogies rather than as a reference text. That is, it has been designed for students to read the examples, work initial problems (checkpoints) to help them check their ability to mimic or identify questions to ask, and to then seek help while working on exercises. It can be easily incorporated into flipped classrooms and asynchronous learning.

Because of the focus on active learning, MiTaL does not provide an identical example to mimic for each question in the exercises. Rather each section provides an simple example, a more complex example, and some explanation about what is important. There are also examples that present the whole thought process involved in figuring out what to do and executing the method. These are designed to illustrate how a person approaches new problems. There are also a few examples on how to use examples to illustrate how a non-identical example can still be used on a problem. Then exercise sections are provided for them to test their ability to recognize and use the mathematics from that section.

The check points (exercises in the reading) are self-grading (using MyOpenMath) with some feedback so the reader can determine what if any questions they need to ask. Videos, where included, are presentations of the introduction of the concept in that section. Homework by default is live, online problems that provide some feedback. The scores on these cannot be saved in an LMS however. If you wish to use MyOpenMath, the problems, and even a shell, can be provided. Contact the author for information.

Other formats are available. A PDF version, intended for use on devices (as opposed to print), is available from the website. Homework is, naturally, not live. Other formats and even other versions are available, because the text is an open educational resource. Contact the author for access to the PreTeXt source.

The projects are an integral part of the general education goals. These are intended to be assigned after relevant material is covered. They require students to use topics from that chapter to perform calculations and then interpret the results of their work. This is part of achieving general education outcomes. The projects also provide a opportunity for students to express calculations using standard mathematical notation and to communicate mathematical results in clear language.

For context, here is a brief history. The first version of this text was written to transition to an OER for MATH A104 Technical Mathematics at the Univesity of Alaska Anchorage. Commercially available texts emphasized memorizing problem types with limited critical thinking. As such they could not be used to satisfy the general education requirement. There was also a desire to reduce costs including for textbooks

and online homework systems. As a result this book was created with matching homework in MyOpenMath.

Served disciplines at UAA included auto/diesel, heavy equipment mechanics, welding and non-destructive testing, aviation mechanics, piloting, air traffic controll, and medical certification programs.

The general education outcome around which this was designed was: Quantitative courses develop abilities to reason mathematically and analyze quantitative and qualitative data to reach sound conclusions for success in undergraduate study and professional life. The indicators were

- Interprets info in mathematical form (equations, graphs, diagrams, tables, words)
- Represents and/or converts relevant quant info and explain its assumptions and limits
- Applies mathematical forms (equations, graphs, diagrams, tables, words) to quantitative problems to reach sound conclusions
- Communicates quantitative results appropriate to the problem or context

Contents

Acknowledgements v

For Students vii

For Faculty ix

1 Cross Cutting Topics 1

1.1 Units . . . 1
1.2 Accuracy and Precision . . . 16
1.3 Working with Applications . . . 28
1.4 Project: Literal Formula . . . 37

2 Ratios 39

2.1 Percents . . . 39
2.2 Mixtures . . . 49
2.3 Ratios . . . 54
2.4 Proportions . . . 61
2.5 Medical Ratios . . . 74
2.6 Project: False Position . . . 83
2.7 Project: Arclength Estimation . . . 84

3 Models 87

3.1 Linear Expressions . . . 87
3.2 Representing Data . . . 99
3.3 Identifying Rates . . . 131
3.4 Variation . . . 141
3.5 Rational Expressions . . . 149
3.6 Linear Systems . . . 159
3.7 Project: Radiation Dosage . . . 168
3.8 Project: Biking in Kansas and Alaska . . . 170
3.9 Project: Constraints on Dilution Problems . . . 172

4 Geometry 175

4.1 Geometric Reasoning 2D . . . 175

4.2 Geometric Reasoning 3D . . . 201
4.3 Project: Unicorn Cake . . . 228
4.4 Project: Watering Grass . . . 229
4.5 Project: Order Flooring . . . 232
4.6 Project: Rounded Rectangle . . . 234

5 Quadratics **235**

5.1 Quadratics . . . 235
5.2 Solving Quadratics . . . 239
5.3 Roots . . . 245
5.4 Graphs of Quadratics . . . 247
5.5 Project: Approximation of Sums . . . 264

6 Exponential **267**

6.1 Exponential Relations . . . 267
6.2 Graphs of Exponential Functions . . . 277
6.3 Logarithm Properties . . . 294
6.4 Solving Equations Using Logarithm . . . 303
6.5 Project: Time of Death . . . 312

7 Trigonometry **315**

7.1 Trigonometric Ratios . . . 315
7.2 Inverse Trigonometric Relationships . . . 333
7.3 Non-Right Triangles . . . 339
7.4 Sine Wave Properties . . . 353
7.5 Project: Effects of Scale on Error . . . 363
7.6 Project: A Clever Sundial . . . 365
7.7 Project: Wind Correction Angle . . . 366

Chapter 1

Cross Cutting Topics

1.1 Units

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Unit conversion (skill)
- Set up and solve proportions (skill)

This book is designed to present mathematics in various contexts including a variety of trades. As a result numbers will frequently be connected to units such as length (feet, meters), time (seconds) or others. These units are part of the arithmetic, and so we must learn what they mean and how to perform this arithmetic. Sometimes the units even suggest to us how to setup the arithmetic.

In this section we will introduce the units so we can understand them when we read technical material (Item 2) and use them correctly to communicate results to others (Item 3). We will also learn to convert units (Item 2.d) which involves our first use of proportions (Item 2.a).

We measure many things such as distance, time, and weight. We describe these measurements in terms of units like miles, hours, and pounds. But have you ever stopped to think about how these units are defined?

The story of some of these units is lost in history. For example dividing the day into 24 units began with ancient Egyptians. They did not record, that we know of, the reason for choosing 24 units as opposed to 30 or any other number.

Other units, such as the metric (also called the International System of Units or simply SI) are much more modern. Initially many units were based on something physical. For example, one calorie is the amount of heat it takes to raise the temperature of one gram of water 1° C. The meter was originally defined as one ten-millionth of the distance from the equator to the north pole. The problem with this type of measurement is that it is neither fixed (depends on where on the equator your begin) nor easy to measure.

Thus modern definitions were developed. The length of a meter was changed to mean the length of a bar of metal kept in special storage in France. The bar had been carefully constructed and was used to confirm other measurement devices were correctly calibrated. It was changed yet again to be based on wavelengths of radiation. These are uniform no matter where they are done, so they can be used by many people to construct simple measurement tools.

1.1.1 Types of Measurement

First we will look at units for different types of measurement in the two major systems. In a specific trade you will need to memorize the units you use most often. For this class you should ask your instructor which units must be memorized and which you may look up when working on problems.

Note that the U.S. Customary system (related to the British Imperial system) is non-uniform, so there are multiple names for some types of measurements. This is in contrast to the SI (metric) which has one name for each property and prefixes to indicate the size. Table 1.1.1 lists names of units for both systems. It is important to be able to recognize which unit (name) goes with which type of measurement (e.g., length, volume, …).

Table 1.1.1 Units of Measure

Measuring	US Customary	Metric
Length	inch (in) foot (ft) yard (yd) mile (mi) nautical mile (nm)	meter (m)
Volume	fluid ounce (oz) cup (c) pint (pt) quart (qt) gallon (g)	liter (L or ℓ)
Weight	ounce (oz) pound (lb)	gram (g)
Temperature	degrees Fahrenheit (F)	degrees Celsius (C)
Pressure	inches of mercury (inHg) pounds per square inch (psi)	bar Pascal (Pa)
Time	second (s) minute (min) hour (hr)	

One name: two meanings Note that fluid ounces and weight ounces are not the same unit. Ten (10) fluid ounces of milk does not weigh ten (10) ounces. You must determine which ounce is referenced by the context. This can be tricky in recipes, which is a good reason to use SI units.

Note a gram is a unit of mass rather than than a unit of weight. Pound and ounce on the other hand are units of weight. Nevertheless gram is often used to describe weight because it is easy to switch between it and weight. Namely, mass can be obtained by dividing by the acceleration due to gravity (see a physics book for more information). The official unit for weight (a force) is a Newton, but we will not use that in this book.

1.1.2 U.S. Customary

Often we need to convert between units within the U.S. Customary system. This section provides the information needed for conversion and examples of performing them. It is an example of units suggesting how we setup the calculation.

Why would we need to convert units? This can occur because measurements were taken with different scales. For instance, we cannot add 3 inches to 2.2 feet without changing one to make the units match.

Why are there multiple units in the first place? There are different units for different scales (e.g., inches for small lengths and miles for long distances). This is a result of the U.S. Customary system being developed from the British Imperial system which was based on disparate measurements from multiple centuries ago (look up unit names in an etymological dictionary for fun). Converting between units therefore requires remembering special numbers for conversion. Most of these you likely know.

Table 1.1.2 Converting within U.S. Customary

Measuring	Unit 1	Unit 2
Length	1 nm	6076 ft
	1 mi	5280 ft
	1 yd	3 ft
	1 ft	12 in
Area	1 acre	43,560 ft^2
Volume	1 g	4 qts
	1 qt	2 pts
	1 pt	2 c
	1 c	8 oz
	1 g	0.1337 in^3
Weight	1 ton	2000 lbs
	1 lb	16 oz
Time	1 year	365 days
	1 day	24 hrs
	1 hr	60 mins
	1 min	60 secs

Of course a year is not always the same number of days. In each circumstance it is important to determine whether we can use the common approximation of 365 days without injury or loss.

Review each of these examples to see how to convert from one U.S. Customary unit to another.

Example 1.1.3 How many quarts is 2.3 gallons?

From Table 1.1.2 we know (or can look up) that each gallon is 4 quarts. This means we have $\frac{4 \text{ quarts}}{1 \text{ gallon}}$. This suggests that we can multiply 2.3 by this ratio, because the gallons will divide out.

$$2.3 \text{ gallons} \cdot \frac{4 \text{ quarts}}{\text{gallon}} =$$
$$\frac{2.3 \cancel{\text{ gallons}}}{1} \cdot \frac{4 \text{ quarts}}{\cancel{\text{gallon}}} =$$
$$2.3 \cdot 4 \text{ quarts} = 9.2 \text{ quarts}$$

We can be confident we set up the conversion correctly because the only units left are quarts. ▲

Sometimes to convert units we need to convert more than once.

Example 1.1.4 How many cups is 1.5 quarts?

While we do not have an entry for cups per quart in the conversion table, we do have entries for quarts to pints and pints to cups. This suggests we can put these two conversions together to convert quarts to cups.

$$1.5 \text{ quarts} \cdot \frac{2 \text{ pints}}{\text{quart}} =$$
$$\frac{1.5 \cancel{\text{ quarts}}}{1} \cdot \frac{2 \text{ pints}}{\cancel{\text{quart}}} =$$
$$1.5 \cdot 2 \text{ pints} = 3.0 \text{ pints}.$$

Now we can convert the pints to cups.

$$3.0 \text{ pints} \cdot \frac{2 \text{ cups}}{\text{pint}} =$$
$$\frac{3.0 \cancel{\text{ pints}}}{1} \cdot \frac{2 \text{ cups}}{\cancel{\text{pint}}} =$$
$$3.0 \cdot 2 \text{ cups} = 6.0 \text{ cups}.$$

When we know that we need multiple conversions we can calculate them all at once.

$$1.5 \text{ quarts} \cdot \frac{2 \text{ pints}}{\text{quart}} \cdot \frac{2 \text{ cups}}{\text{pint}} = 6.0 \text{ cups}.$$

When we use more than one conversion, we are careful to set up the ratios so that the intermediate units (quarts and pints in this case) are divided out leaving us with only the desired unit. ▲

Nothing restricts this process to two at a time.

Example 1.1.5 How many cups is 1.7 gallons?

Solution. The conversion table does not include the number of cups per gallon. However, we can start by converting gallons to quarts. Then looking at the table again, we can convert quarts to pints, and finally we can convert pints to cups.

As with the previous example we can treat each conversion as a ratio of units. We setup the units so that multiplying will result in units dividing out.

$$1.7 \text{ gallons} \cdot \frac{4 \text{ quarts}}{\text{gallon}} \cdot \frac{2 \text{ pints}}{\text{quart}} \cdot \frac{2 \text{ cups}}{\text{pint}} = 27.2 \text{ cups}$$

▲

Example 1.1.6 How many days is 17 hours?

Solution. Here we are going from a small unit (hours) to a bigger one (days). This does not change our process. We can still multiply the amount by the unit conversion. Because we want to end up with days we use $\frac{1 \text{ day}}{24 \text{ hours}}$

$$17 \text{ hours} \cdot \frac{1 \text{ day}}{24 \text{ hours}} \approx 0.7083 \text{ days}$$

▲



Standalone
In Context

Check that you can perform a unit conversion using this Checkpoint.

Checkpoint 1.1.7 Convert the measurement. You may find it useful to use this table.
48 cups = ____ gal

1.1.3 Metric (SI)

Just as with U.S. Customary units, we often need to convert between SI units. This section provides the information needed for conversion and examples of performing them. Again, units suggest how we setup the calculations.

Rather than have different names for different scales, metric uses one name of the unit (e.g., liter) and then uses prefixes to indicate size. These can be converted easily, because each prefix is a power of ten (uniform).

You will need to memorize a few of the prefixes. As with units which ones depends on your work. Ask your instructor which prefixes you should memorize for this course.

Table 1.1.8 Metric Prefixes

Multiple	Prefix
10^{12}	tera (T)
10^{9}	giga (G)
10^{6}	mega (M)
10^{3}	kilo (k)
10^{2}	hecto (h)
10	deka (da)
$1/10$	deci (d)
$1/10^{2}$	centi (c)
$1/10^{3}$	milli (m)
$1/10^{6}$	micro (μ)
$1/10^{9}$	nano (n)
$1/10^{12}$	pico (p)

This next table illustrates these prefixes in the context of length (meters). Notice how it is easier to avoid the fractions (last three entries).

Table 1.1.9 Metric Conversion

Scaled Unit	Base Unit
kilometer (km)	$10^3 = 1000$ meters
hectometer	$10^2 = 100$ meters
10 decimeters	meter
$10^2 = 100$ centimeters	meter
$10^3 = 1000$ millimeters	meter

Review each of these examples to see how to convert from one SI unit to another.

Example 1.1.10 How many centimeters is 3.8 meters?

From Table 1.1.8 we know $10^2 = 100$ centimeters is 1 meter, that is, $\frac{100 \text{ cm}}{1 \text{ m}}$. This suggests that we can multiply 3.8 by the ratio which will cause the meters units to divide out.

$$3.8 \text{ m} \cdot \frac{100 \text{ cm}}{\text{m}} =$$
$$\frac{3.8 \cancel{\text{m}}}{1} \cdot \frac{100 \text{ cm}}{\cancel{\text{m}}} =$$
$$3.8 \cdot 100 \text{ cm} = 380 \text{ cm}.$$

Note, because centi is a power of ten (10^2) the result is shifting the decimal place two positions. 3.8 meters becomes 380 centimeters.

Using this idea we can convert 0.76 meters to 76 centimeters by just shifting the decimal (no additional process necessary). ▲

Example 1.1.11 How many kilotons is 2.3 megatons?

We know one kiloton is 10^3 tons and one megaton is 10^6 tons. Because we have two units we can convert twice.

$$2.3 \text{ megatons} \cdot \frac{10^6 \text{ tons}}{\text{megatons}} \cdot \frac{\text{kilotons}}{10^3 \text{ tons}} =$$
$$\frac{2.3 \cancel{\text{megatons}}}{1} \cdot \frac{10^6 \cancel{\text{tons}}}{\cancel{\text{megatons}}} \cdot \frac{\text{kilotons}}{10^3 \cancel{\text{tons}}} =$$

$$\frac{2.3 \cdot 10^6}{10^3} \text{ kilotons} =$$
$$2.3 \cdot 10^3 \text{ kilotons} = 2,300 \text{ kilotons.}$$

Just as in the previous example, we can pay attention to the powers of ten to find a shortcut. Kilo and mega are three powers apart ($6-3=3$), which means we shift the decimal position three places (we see this in the last two steps above). Because we are converting from a large unit to a smaller unit, we move the decimal place to the right (make the number bigger). 2.3 megatons is 2,300 kilotons. ▲

Example 1.1.12 How many centiliters is 13.6 milliliters?

Solution. We know $10^2 = 100$ centiliters is 1 liter and $10^3 = 1000$ milliliters is 1 liter. This means we shift the decimal $2-(3) = -1$. Because we are moving from a smaller unit to a larger unit, we move the decimal place to the left (shown by the negative) to make the number smaller. 13.6 milliliters is 1.36 centiliters. ▲

Checkpoint 1.1.13 Convert the units below. You may find it useful to use this table.
1420 mL = _____ L

1.1.4 Converting between Systems

Commonly we end up with measurements in both U.S. Standard system and SI. We will need to convert all units to one system before using them together. This process is the same as converting one Standard unit to another (e.g., converting miles to feet). This section provides the information needed for conversion and examples of performing them. It is an example of units suggesting how we setup the calculation.

Table 1.1.14 U.S. Customary to SI

Measuring	Standard	SI
Length	1 nm	1.85200 km
Length	1 mi	1.60934 km
	1 ft	0.304800 m
	1 in	2.54000 cm
Volume	1 gal	3.78541 L
	1 oz	29.5735 mL
Weight	1 lb	0.453592 kg
	1 oz	28.3495 g

Table 1.1.15 SI to U.S. Customary

Measuring	SI	Standard
Length	1 km	0.621371 mi
	1 m	3.28084 ft
	1 cm	0.393701 in
Volume	1 L	0.264172 gal
	1 mL	0.0338140 oz
Weight	1 kg	2.20462 lb
	1 g	0.0352740 oz

Review each of these examples to see how to convert between U.S. Customary units and SI units.

Example 1.1.16 How many kilometers is 26.2 miles?

Solution. From Table 1.1.14 we know each mile is 1.60934 km; this means there is

$$\frac{1.60934 \text{ km}}{1 \text{ mi}}.$$

The ratio suggests that we can multiply 23.6 miles by the ratio, because the miles will divide out.

$$26.2 \text{ mi} \cdot \frac{1.60934 \text{ km}}{\text{mi}} = 42.164708 \text{ km} \approx 42.2 \text{ km}$$

We have rounded here to 3 digits to match the original number (26.2). A reason to do so is presented in Section 1.2. ▲

Example 1.1.17 How many inches is 15 centimeters?

Solution. From Table 1.1.15 we know each centimeter is 0.393701 inches; this means there is

$$\frac{0.393701 \text{ in}}{1 \text{ cm}}.$$

The ratio suggests that we can multiply 15 centimeters by the ratio, because the centimeters will divide out.

$$15 \text{ cm} \cdot \frac{0.393701 \text{ in}}{\text{cm}} = 5.905515 \text{ in} \approx 5.9 \text{ in}$$

We have rounded here to 2 digits to match the original number (15). A reason to do so is presented in Section 1.2. ▲

Example 1.1.18 How many inches is 1 meter?

Solution. From Table 1.1.15 we know each meter is 3.28084 feet. From Table 1.1.2 that each foot is 12 inches. We use the method of setting up a product of ratios so that the units divide out. We start with meters, so the first ratio must be feet per meters. We want to end with inches so the second ratio must be inches per feet.

$$1 \text{ m} \cdot \frac{3.28084 \text{ ft}}{1 \text{ m}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 39.37008 \text{ in} \approx 39 \text{ in}$$

We have rounded here to 2 digits to match the original number (15). A reason to do so is presented in Section 1.2. ▲

Checkpoint 1.1.19 While purchasing gas on vacation in Canada, Kiersten wonders how many gallons she is purchasing. You may find it useful to use this table.

66 L = ____ gal

1.1.5 Converting Compound Units

Some units, such as speeds, are compound. For example speed is distance per time. This section provides examples of converting compound units.

Example 1.1.20 How many meters per second is 15 miles per hour?

We start with $\frac{15 \text{ mi}}{1 \text{ hr}}$. We can convert miles to feet and feet to meters (multi-step conversion like Example 1.1.4). The conversion ratios suggest we can multiply the 15 mi/hr by the conversion ratios.

$$\frac{15 \text{ mi}}{1 \text{ hr}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{0.3048 \text{ m}}{1 \text{ ft}} \approx \frac{24140 \text{ m}}{1 \text{ hr}}$$

We can use the same method (multiplying by conversion ratios to divide out units) to also conver hours to seconds.

$$\frac{24140 \text{ m}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \approx 6.706 \frac{\text{m}}{\text{sec}}.$$

Note, we could perform that conversion in on step by multiplying all the conversion ratios at once. ▲

Example 1.1.21 How many pounds does a tablespoon of water weigh? Note that one gallon of pure water weighs 8 lbs. Also a tablespoon is a half fluid ounce.

Solution. We need to convert the gallons into tablespoons. Because conversions are ratios, we multiply 8 lbs by the necessary conversions. The ounces (oz) in this conversion are fluid (volume) ounces: we are not

converting pounds to ounces here.

$$\frac{8 \text{ lbs}}{1 \text{ gal}} \cdot \frac{1 \text{ gal}}{4 \text{ qts}} \cdot \frac{1 \text{ qt}}{2 \text{ pints}} \cdot \frac{1 \text{ pint}}{2 \text{ cups}} \cdot \frac{1 \text{ cup}}{16 \text{ oz}} \cdot \frac{1 \text{ oz}}{2 \text{ tbs}} =$$
$$\frac{8 \text{ lbs}}{1 \cancel{\text{ gal}}} \cdot \frac{1 \cancel{\text{ gal}}}{4 \cancel{\text{ qts}}} \cdot \frac{1 \cancel{\text{ qt}}}{2 \cancel{\text{ pints}}} \cdot \frac{1 \cancel{\text{ pint}}}{2 \cancel{\text{ cups}}} \cdot \frac{1 \cancel{\text{ cup}}}{16 \cancel{\text{ oz}}} \cdot \frac{1 \cancel{\text{ oz}}}{2 \text{ tbs}} =$$
$$\frac{8 \text{ lbs}}{4 \cdot 2 \cdot 2 \cdot 16 \cdot 2 \text{ tbs}} = 0.015625 \frac{\text{lbs}}{\text{tbs}}.$$

▲

Checkpoint 1.1.22 Convert 76 miles per hour to feet per second. You may wish to use this conversion table.

76 miles per hour = _____ feet per second. Round your answer to the nearest tenth.

Another kind of compound unit is square units such as square feet or seconds squared. When converting these we must account for the square.

Example 1.1.23 Convert 2 acres to units of square miles.

Solution. First we note that an acre is $43,560 \text{ ft}^2$. From Table 1.1.2 we know that there are 5280 ft per mile. These conversion ratios suggest that we can multiply the 2 acres by the ratios to obtain the result in square miles.

$$2 \text{ acres} \cdot \frac{43,560 \text{ ft}^2}{\text{acre}} \cdot \frac{\text{mi}}{5280 \text{ ft}} \cdot \frac{\text{mi}}{5280 \text{ ft}} =$$
$$2 \text{ acres} \cdot \frac{43,560 \text{ ft}^2}{\text{acre}} \cdot \left(\frac{\text{mi}}{5280 \text{ ft}}\right)^2 =$$
$$2 \text{ acres} \cdot \frac{43,560 \text{ ft}^2}{\text{acre}} \cdot \frac{\text{mi}^2}{5280^2 \text{ ft}^2} = \frac{1}{320} \text{ mi}^2$$
$$= 0.003125 \text{ mi}^2$$

It is not necessary to write all of the steps above if you understand how the final conversion line is obtained. The steps are included here to show how the squares show up in the final conversion. ▲

Checkpoint 1.1.24 Convert 314 square inches to square feet. Round your answer to the nearest hundredth.

_____ ft^2





Standalone
In Context

Example 1.1.25 Calculate the value and resulting (simplified) units for

$$\frac{1}{2} \cdot \frac{0.002309 \text{ slugs}}{\text{ft}^3} \cdot (174 \text{ ft}^2)(3.1067) \left(\frac{75.95 \text{ ft}}{\text{s}}\right)^2.$$

Note, the unit slug is defined as

$$\text{slug} = \frac{\text{lbs s}^2}{\text{ft}}.$$

$$\begin{aligned}
\frac{1}{2}\cdot\frac{0.002309\ \text{slugs}}{\text{ft}^3}\cdot(174\ \text{ft}^2)(3.1067)\left(\frac{75.95\ \text{ft}}{\text{s}}\right)^2 &= \text{First, substitute slugs}\\
\frac{1}{2}\cdot\frac{0.002309\ \text{lbs s}^2}{\text{ft}\cdot\text{ft}^3}\cdot(174\ \text{ft}^2)(3.1067)\left(\frac{75.95\ \text{ft}}{\text{s}}\right)^2 &= \text{Second, handle powers}\\
\frac{1}{2}\cdot\frac{0.002309\ \text{lbs s}^2}{\text{ft}^4}\cdot(174\ \text{ft}^2)(3.1067)\left(\frac{75.95\ \text{ft}^2}{\text{s}^2}\right) &= \text{Now, collect units}\\
\frac{1}{2}(0.002309)(174)(3.1067)(75.95)\frac{\text{lbs s}^2\ \text{ft}^2\ \text{ft}^2}{\text{ft}^4\text{s}^2} &= \text{Combine units}\\
\frac{1}{2}(0.002309)(174)(3.1067)(75.95)\frac{\text{lbs s}^2\ \text{ft}^4}{\text{ft}^4\text{s}^2} &= \text{Calculate the result}\\
3599.963183\frac{\text{lbs s}^2\ \text{ft}^4}{\text{ft}^4\text{s}^2} &= \text{Divide units}\\
3599.963183\frac{\text{lbs}\ \cancel{\text{s}^2}\ \cancel{\text{ft}^4}}{\cancel{\text{ft}^4}\ \cancel{\text{s}^2}} &= \text{Divide units}\\
3599.963183\ \text{lbs} &= \text{Round using sigfigs}\\
3600\ \text{lbs}
\end{aligned}$$

▲

1.1.6 Exercises

Dimensional Analysis. Identify the correct units to make these work.

1. Select the correct units to complete the conversion below.
gallons $*$ _____ = miles
Answer:

1. miles/gallons
2. gallons
3. 1/gallons
4. miles
5. gallons/miles
6. 1/miles

2. Select the correct units to correctly complete the calculation below.
(people)$*$(_____) = (dollars)
Answer:

1. people
2. people/dollars
3. dollars
4. 1/dollars

5. dollars/people

6. 1/people

Converting Units. Convert the units as directed. Use the conversion tables (Table 1.1.2, Table 1.1.9, or Table 1.1.14) as needed.

3. Convert the measurement. You may find it useful to use this table.
48 cups = ____ gal

4. Convert $4\frac{11}{12}$ hours to minutes. Enter your answer as an integer or a reduced fraction.
$4\frac{11}{12}$ hours =_____ minutes

5. Convert the measurement.
6 days = _____ sec

6. Convert 76 miles per hour to feet per second. You may wish to use this conversion table.
76 miles per hour = _____ feet per second. Round your answer to the nearest tenth.

7. Convert 332 square inches to square feet.
Round your answer to the nearest hundredth.
_____ square feet

8. 179 in. to yards, feet, and inches
__ yds __ ft __ in

9. Convert the measurement
2610 mL = _____ L

10. Convert the measurement
2.1 g = _____ mg

11. Convert the measurement
7 kg = _____ g

12. Convert the measurement
5 m = _____ cm

13. Convert the measurement
433 cm = _____ m

14. Convert the measurement using the rules of SI prefixes.
2.5 terabyte = _____ gigabyte

15. Convert 6.7 square meters to square centimeters.
_____ square centimeters

16. A small rectangular panel measures 3.4 cm by 3.2 cm. What is its area in square millimeters?
_____ square millimeters

17. Convert 375.97 square yards into square meters. Round your answer off to the nearest *hundredth* if necessary.
_____ square meters

U.S. Customary System

1 yd = 3 ft	1 ft = 12 in	1 mi = 5280 ft	1 acre = 43,560 ft^2
1 lb = 16 oz	1 ton = 2000 lb	16 T = 1 cup	1 cup = 8 fluid oz
1 qt = 4 cups	1 gal = 4 qt	1 gal $\approx$ 231 in^3	1 $ft^3 \approx$ 7.48 gal

Metric System

1 m = 1000 mm	1 m = 100 cm	1 cm = 10 mm	1 m = 1,000,000 microns (μ)
1 km = 1000 m	1 kg = 1000 g	1 g = 1000 mg	1 kL = 1000 L
1 L = 1000 mL	1 cm^3 = 1 mL	1 m^3 = 1000 L	1 hectare (ha) = 10,000 m^2

Conversion between the U.S. Customary and Metric Systems

1 in = 2.54 cm	1 yd ≈ 0.914 m	1 m ≈ 39.37 in	1 m ≈ 3.281 ft
1 mi ≈ 1.609 km	1 lb ≈ 453.6 g	1 oz ≈ 8.35 g	1 lb ≈ 4.448 N
1 kg ≈ 2.205 lb	1 L ≈ 1.057 qt	1 ft^3 ≈ 28.32 L	1 fluid oz ≈ 29.574 mL

18. Convert 178.7 miles per hour into feet per minute. Round your answer off to the nearest *hundredth* if necessary.

_____ feet per minute

U.S. Customary System

1 yd = 3 ft	1 ft = 12 in	1 mi = 5280 ft	1 acre = 43,560 ft^2
1 lb = 16 oz	1 ton = 2000 lb	16 T = 1 cup	1 cup = 8 fluid oz
1 qt = 4 cups	1 gal = 4 qt	1 gal ≈ 231 in^3	1 ft^3 ≈ 7.48 gal

Metric System

1 m = 1000 mm	1 m = 100 cm	1 cm = 10 mm	1 m = 1,000,000 microns (μ)
1 km = 1000 m	1 kg = 1000 g	1 g = 1000 mg	1 kL = 1000 L
1 L = 1000 mL	1 cm^3 = 1 mL	1 m^3 = 1000 L	1 hectare (ha) = 10,000 m^2

Conversion between the U.S. Customary and Metric Systems

1 in = 2.54 cm	1 yd ≈ 0.914 m	1 m ≈ 39.37 in	1 m ≈ 3.281 ft
1 mi ≈ 1.609 km	1 lb ≈ 453.6 g	1 oz ≈ 8.35 g	1 lb ≈ 4.448 N
1 kg ≈ 2.205 lb	1 L ≈ 1.057 qt	1 ft^3 ≈ 28.32 L	1 fluid oz ≈ 29.574 mL

19. Conver 159.1 grams per square centimeter into ounces per square inch. Round your answer off to the nearest *hundredth* if necessary.

_____ ounces per square inch

U.S. Customary System

1 yd = 3 ft	1 ft = 12 in	1 mi = 5280 ft	1 acre = 43,560 ft^2
1 lb = 16 oz	1 ton = 2000 lb	16 T = 1 cup	1 cup = 8 fluid oz
1 qt = 4 cups	1 gal = 4 qt	1 gal = 231 in^3	1 ft^3 = 7.48 gal

Metric System

1 m = 1000 mm	1 m = 100 cm	1 cm = 10 mm	1 m = 1,000,000 microns
1 km = 1000 m	1 kg = 1000 g	1 g = 1000 mg	1 kL = 1000 L
1 L = 1000 mL	1 cm^3 = 1 mL	1 m^3 = 1000 L	1 hectare = 10,000 m^2

Conversion between the U.S. Customary and Metric Systems

1 in = 2.54 cm	1 yd = 0.914 m	1 m = 39.37 in	1 m = 3.281 ft
1 mi = 1.609 km	1 lb = 453.6 g	1 oz = 28.35 g	1 lb = 4.448 N
1 kg = 2.205 lb	1 L = 1.057 qt	1 ft^3 = 28.32 L	1 fluid oz = 29.574 mL

20. How many millimeters are there in a meter? _____

How many liters are in a decaliter? _____

How many centigrams are there in a gram? _____

Which prefix indicates a bigger quantity? hecto
kilo

Which prefix indicates a bigger quantity? deca
deci

Which prefix indicates a bigger quantity? mega
kilo

Which prefix indicates a bigger quantity? milli
centi

Applications. Answer these questions by converting units as needed.

21. Taylor mixed 4 gallons of lemonade and poured it into three 5-quart jugs. How many cups of lemonade were left over after she filled the jugs?

This table may be useful for this problem.

___ cups

22. A corn stalk grew 3 inches in the first month after it was planted, and since then it grew another 5 feet.

What is the total height of the corn in feet and inches? _____ ft _____ in

What is the to total height of the corn stalk in inches? _____

What is the total height of the corn stalk in feet? _____ Round your answer to 2 decimal places.

23. You are in charge of drinks for a community barbecue. You need to supply at least 120 cups of beverage to provide enough for the projected number of people that will attend. So far, you have received the following donations:

- Enough mix to make 4 gallons of lemonade
- 5 bottles of fruit juice that each contain 64 fl. oz.

How many cups of beverage do you have?

____

Will you have enough for the barbecue?

1. yes
2. no

24. Jean's bedroom is 11 feet by 12 feet. She has chosen a carpet which costs $33.85 per square yard. This includes installation. Assuming she can buy partial square yards and wastes no carpet, determine her cost to carpet her room. Round results to the nearest cent.

$ _____

How much would she have saved if she went with the carpet that costs $31.30 per square yard instead?

$ _____

25. Sahana is making a quilt and she has determined she needs 670 square inches of gray fabric and 1058 square inches of blue. How many square yards of each material will she need to purchase from the fabric store?

The store only sells fabric by the by the quarter yard.

The gray fabric: _____ square yards

The blue fabric: _____ square yards

How many total yards of fabric will she have to buy?

_____ square yards

26. A unit of measure sometimes used in surveying is the *link;* 1 link is about 8 inches. About how many links are there in 3 feet? Do not round your answer.

There are ___ links in 3 feet.

27. David has 11 yd. of material that he will cut into strips 7 in. wide to make mats. How many mats can David make?

___

28. I saw a job opening for a Graphic Designer that pays $43,000 per year salary. Assume that a regular work week for this job is 30 hours, and that you will work 48 weeks in a year.

What is the hourly pay for this job? We will answer the question by converting *$43,000 per year* into *dollars per hour*.

If we begin with the fraction $\frac{\$43000}{\text{year}}$, we can multiply by two unit fractions to complete the conversion. What are these fractions? Choose the correct fractions in the calculation below:

$\frac{\$43000}{\text{year}} \times$

1. 48 weeks/1 year
2. 48 years/1 week
3. 1 week / 48 years
4. 1 year/48 weeks

$\times$

1. 1 hour/30 weeks
2. 1 week/30 hours
3. 30 hours/1 week
4. 30 weeks/1 hour

Finally, what is the hourly pay for this job?
Hourly pay = _____ dollars/hour

29. Part 1 of 2

Ross was driving at 97 feet per second on the freeway the other day. If the speed limit is 70 miles per hour, was he driving too fast? Answer the question by converting 97 feet per second into miles per hour.

To answer this question, we will convert the numerator into miles and the denominator into hours.

If we begin with the fraction $\frac{97 \text{ feet}}{1 \text{ second}}$, we can multiply by three unit fractions to complete the conversion. What are these fractions? Choose the correct fractions in the calculation below:

$\frac{97 \text{ feet}}{1 \text{ second}} \times$

1. 5280 ft/1 mi
2. 5280 mi/1 ft
3. 1 ft/5280 mi
4. 1 mi/5280 ft

$\times$

1. 1 sec/60 min
2. 1 min/60 sec
3. 60 min/1 sec
4. 60 sec/1 min

$\times$

1. 1 hr/60 min
2. 1 min/60 hrs
3. 60 min/1 hr

4. 60 hrs/1 min

Part 2 of 2
$\frac{97 \text{ feet}}{1 \text{ second}} \times \frac{1 \text{ mi}}{5280 \text{ ft}} \times \frac{60 \text{ sec}}{1 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hr}}$
Was Ross driving too fast?
Answer:

1. No
2. Yes

_____70

30. Part 1 of 38.92 $\frac{\text{dollars}}{\text{yard}^2}$Note about this unityards^{2}Aurora is sure she saw this advertised online for $0.55\frac{\text{cents}}{\text{inch}^2}$, and she wants to know if it is a better deal in the store or online. She will take a minute to convert the price from $\frac{\text{dollars}}{\text{yard}^2}$ to $\frac{\text{cents}}{\text{inch}^2}$ to see which is the better deal.

Aurora knows the following facts:

- 1 yard = 3 feet
- 1 ft = 12 inches
- 1 dollar = 100 cents

$\frac{8.92 \text{ dollars}}{\text{yard}^2}$*

1. 1ft / 3yd
2. 1yd / 3ft
3. 3yd / 1ft
4. 3ft / 1yd

*

1. 3yd / 1ft
2. 1yd / 3ft
3. 1ft / 3yd
4. 3ft / 1yd

*

1. 12in / 1ft
2. 1in / 12ft
3. 1ft / 12in
4. 12ft / 1in

*

1. 12in / 1ft
2. 1ft / 12in

3. 12ft / 1in
4. 1in / 12ft

*

1. 100cents / 1dollar
2. 1dollar / 100cents
3. 100dollars / 1cent
4. 1cent / 100dollars

Part 2 of 3

$$\frac{8.92 \text{ dollars}}{\text{yard}^2} * \frac{1 \text{ yd}}{3 \text{ ft}} * \frac{1 \text{ yd}}{3 \text{ ft}} * \frac{1 \text{ ft}}{12 \text{ in}} * \frac{1 \text{ ft}}{12 \text{ in}} * \frac{100 \text{ cents}}{1 \text{ dollar}}$$

What is the final result, rounded to two decimal places?

Answer: _____ $\frac{\text{cents}}{\text{inch}^2}$

Part 3 of 3

Answer:

1. Online
2. The Store

31. A bottle of Vitamin E contains 100 soft gels, each containing 15 mg of vitamin E. How many total grams of vitamin E are in this bottle?

There are __ grams of Vitamin E in the bottle.

32. A battery is rated for 22 amp hours. If the aircraft is using 41 amps, how long in minutes would the battery last? _____

Round down to the nearest minutes. One amp hour means the battery can provide one amp for one hour (unit is 1 amp $\times$ hour).

Quantitative Literacy. Use your knowledge of these items to decide on appropriate units.

33. Select the unit that best fits the scenario.

The box of ice cream contains 4

1. fluid ounce(s)
2. cup(s)
3. gallon(s)

34. Add the following weights:

17 lb 15 oz + 23 lb 9 oz + 27 lb 13 oz

____ pounds ____ ounces

35. Add the following lengths:

15 ft 1 in + 30 ft 4 in + 20 ft 7 in

____ ft ____ in

36. Select the unit that best fits the scenario

My car weighs 800

1. milligram(s)
2. gram(s)
3. kilogram(s)

4. centimeter(s)
5. meter(s)
6. liter(s)
7. milliliter(s)

37. Select the unit that best fits the scenario
The bucket of sand weighs 4

1. milligram(s)
2. gram(s)
3. kilogram(s)

38. Select the unit that best fits the scenario
The perfume bottle holds 20

1. milliliter(s)
2. liter(s)
3. centimeter(s)
4. kilogram(s)
5. meter(s)
6. kilometer(s)

1.2 Accuracy and Precision

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Rounding (skill)
- Significant Figures (skill)
- Determining appropriate rounding from context (critical thinking)

While calculating devices will produce a lot of decimal places, these are not always meaningful nor useful. This section presents different purposes for rounding numbers, examples of using each one, and examples of interpreting numbers we consume in work and life.

First, we will consider what lots of decimal places do and do not mean which will lead to definitions, then we will present a method for reasonably tracking precision, then consider other motivations and matching methods for rounding, and later consider the importance of managing error.

1.2.1 Explanation

When working with measurements, we care about the reasonableness of the results. Suppose four people measure the length of a piece of wood and come up with 1.235 m, 1.236 m, 1.237 m, and 1.234 m. We might conclude that we are confident it is 1.23 m long but we are not certain about the millimeters position. This leads to the concepts of accuracy and precision.

Definition 1.2.1 Accuracy. The **accuracy** of a measurement is how close the measurement is to the actual value. ◆

Example 1.2.2 If the board referenced above is actually 1.2364 m long then all four measurements are accurate to the second decimal place. The second measurement (1.236 m) is accurate to the third decimal place. ▲

Example 1.2.3 Note $\frac{22}{7} \approx 3.142857$ is an approximation for π. Because π to six decimals places (not rounded) is 3.141592, the approximation 22/7 is accurate to only 2 decimal places (i.e., 3.14).

Note, π is not a measurement, rather it is defined theoretically. Thus we can produce an approximation that is as accurate as we have time and will to do. If curious, ask the nearest calculus instructor for details. ▲

Note, if we are measuring something, it is because it is not possible to know the actual value. In the example of measuring the board all we can do is use measuring tools and our use of all such tools has a margin of error. The actual length of the board is a mystery. Because of this we cannot determine the exact value of many kinds of data nor determine how accurate our measurement is. Instead we will settle for repeatability. If we get the same result often enough, we can convince ourselves that it is accurate.

Definition 1.2.4 Error. The **error** of a measurement is the difference between the reported measurement and the actual value. ◆

Example 1.2.5 The number of people at an outdoor concert was 2453. If someone estimated that the number of people was 2500, then that estimate is accurate to 1000's place, but has an error of only $2500 - 2453 = 47$. ▲

Definition 1.2.6 Precision. The **precision** of a measurement is the size of the smallest unit in it. ◆

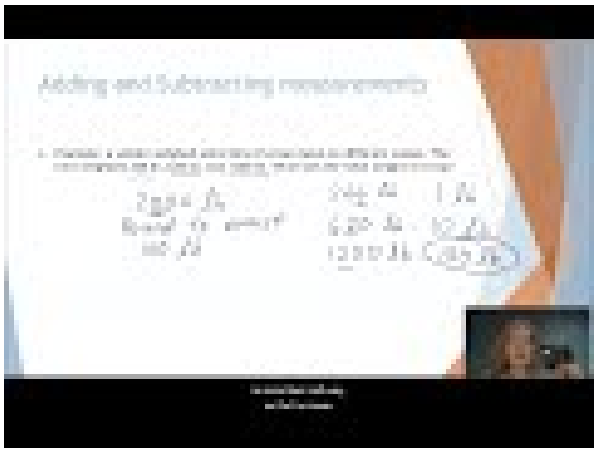
Note we can have high precision with low accuracy. That is, just because we write a lot of decimal places does not mean that number is close to the actual value of the measurement.

Example 1.2.7 The answer to a homework question is 5.7632. If a response given is 5.7647 what are the precision and accuracy of the response?

Precision is effectively the number of decimal places. This is precise to 4 decimal places (the 10,000th position).

Because the response matches the actual value to 2 decimal places (the 100ths position), it is accurate to 2 decimal places.

Because $5.7647 - 5.7632 = 0.0015$, the error is 0.0015. ▲

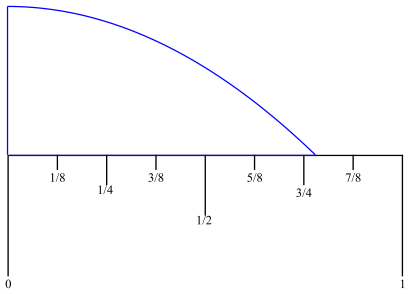


Standalone
In Context

Figure 1.2.8 Introduction to Precision and Accuracy

How do we end up with parts of measurements which are not accurate? Consider the following.

Example 1.2.9



When measuring the width of the blue, curved shape using the ruler (measurement in inches), it is clearly longer than 3/4'' and less than 7/8''. The right side of the shape appears to be less than half way between 3/4'' and 7/8''. Because it appears to be closer to 3/4'', we can state the width is 3/4''. Because the ruler does not have finer markings (e.g., 16ths or 32nds), we cannot be more precise.

We know this measurement is accurate to the nearest 1/8'', because the ruler has those marked and, in this case, we can be confident it is closer to the left side.

To estimate the error we note that the right edge is less than half way between the markings. Half way would be 13/16'' or 1/16'' farther. Thus we can state that the shape is 3/4'' wide with an error that is less than 1/16''. ▲

While other tools for measurement can be more precise, every tool has a limit to its precision similar to this example. We should always be aware of the limitations of measurements when we use them.

1.2.2 Significant Figures

It makes no sense to write numbers that are more precise than they are accurate. For example writing 3.142857 (from the approximation $\pi \approx \frac{22}{7}$ Example 1.2.3) makes no sense, because it is only accurate to the hundreds position (i.e., 3.14). It also makes no sense to perform arithmetic on digits that are not accurate. This section presents a reasonable way of tracking meaningful precision and rounding to maintain it. This will be used in many of the problems involving data from measurments for the rest of the text.

When writing down measurements we need a way to indicate how precise the measurement is. **Significant digits**, also called **significant figures** or simply "sig figs", are a way to do this.

The rules for writing numbers with significant digits have two parts: non-zero digits, and zero digits.

1. All non-zero digits are signficant.
2. Zeros between non-zeros are significant.
3. Any zeros written to the right of the decimal point are significant.
4. If zeros between non-zero digits (on left) and the decimal point (on right) are supposed to be significant, a line is drawn over top of the last significant digit.
5. If the zero to the left of the decimal is significant, the decimal point may be used with no digits to the right (the bar is an easier to read choice however)
6. For numbers less than 1, zeros between the decimal point (on left) and non-zero digits (on right) are not significant.

We can summarize these rules as: write only digits that you mean, and if it is ambiguous, clarify.

Significant digits apply to numbers resulting from measurements. That is, they apply when there is doubt about the accuracy of the number. These will be mixed with exact numbers (numbers with infinite precision). For example the 1/2 in the area of a triangle (Area $= 1/2bh$) is an exact number.

Example 1.2.10 Writing Significant Digits. Each of these numbers is written with five (5) significant digits.

- 10267
- 1.2400
- $7201\overline{0}$
- 53010.
- 2834100
- 0.0010527

▲

Checkpoint 1.2.11 How many significant digits does 306 have? ___
How many significant digits does $4,7\overline{0}0$ have? ___
How many significant digits does 0.00662 have? ___

We also need rules for arithmetic with significant digits. These are based on two principles

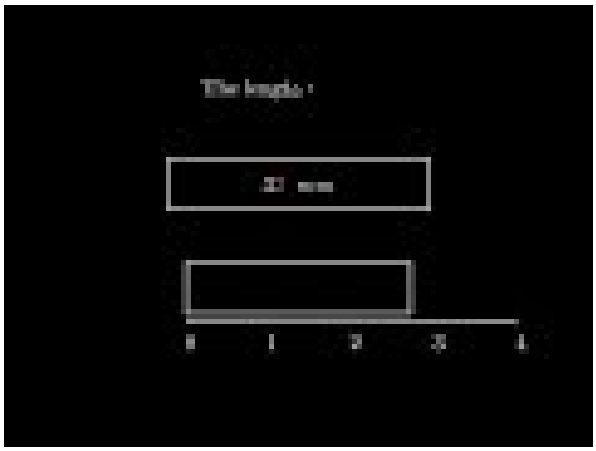
- A result of arithmetic cannot be more precise than the least precise measurement.
- Arithmetic does not increase accuracy.

For addition and subtraction the result (sum or difference) has the same precision as the least precise number added or subtracted. After adding or subtracting we round to the farthest left of the last significant digits.

Example 1.2.12 Subtraction with Significant Digits. $11050 - 723 = 10327 \approx 10330$. The last significant digit of 11050 is in the 10's position (the 5). The last significant digit of 723 is in the 1's (unit) position (the 3). Because 11050 is less precise than 723, the last significant digit of the result is in the 10's position. We do not know the 1's position of 11050, so we cannot know the 1's position in the result. ▲

Example 1.2.13 Addition with Significant Digits. $311 + 8,310 + 202,200 = 210,821 \approx 210,800$. This is rounded to the hundreds position because the least (farthest left) significant digit of the three numbers is in the 100's position (in 202,200). The extra precision of the other two numbers is lost because we don't know to what they should be added. ▲

The significant digits addition/subtraction rule basically says that adding precise data to imprecise data does not increase the precision of the imprecise data. For those who are curious, an explanation of why this rule works is in this video.



Standalone
In Context

Exact numbers may be mixed in calculations with addition/subtraction. For example suppose we are converting temperature from Fahrenheit to Celsius based on a thermometer reading. The formula is $C = \frac{5}{9}(F - 32)$. The 32 and 5/9 are exact numbers (part of the definition of the Fahrenheit and Celsius systems). The F (measured temperature) would have limited precision and therefore determine the precision of the result.

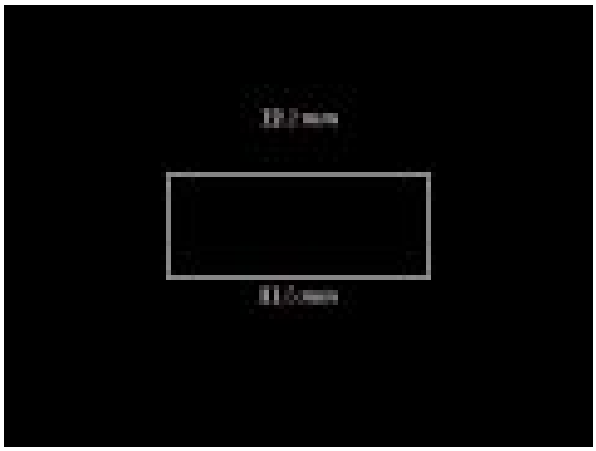
Checkpoint 1.2.14 Calculate each of the following.
$217 + 22.38 + 420 =$ _____
$22.38 - 34 =$ _____

For multiplication and division the result (product or quotient) has the same number of significant digits as the least number of signficant digits of the input numbers.

Example 1.2.15 Division with Significant Digits. $11050/722 \approx 15.30470914 \approx 15.3$. This is rounded to three (3) significant digits because 722 has only 3 significant digits. The fourth significant digit in 11050 (the 5) would be multiplied by the unknown digit after the 3 in 15.3, so we do not know what that would be. ▲

Example 1.2.16 Multiplication with Significant Digits. $17 \times 14\bar{0} \times 3.178 = 7563.64 \approx 7600$. This is rounded to two (2) significant digits because 17 has only two significant digits. The extra precision of the other two numbers is lost. ▲

The significant digit multiplication/division rule basically says that digits that were multiplied by imprecise data cannot be precise. For those who are curious, an explanation of why this rule works is in this video.



Standalone
In Context

Exact numbers may be mixed in calculations with multiplication and division. The following example illustrates how we determine the resulting number of significant digits when exact numbers are mixed with measurements.

Example 1.2.17 Suppose we are converting temperature from Fahrenheit to Celsius based on a thermometer reading. The formula is $C = \frac{5}{9}(F - 32)$. The 32 and 5/9 are exact numbers (part of the definition of the Fahrenheit and Celsius systems).

If we read the temperature as 44.7° F then the conversion is as follows.

$$\begin{aligned} C &= \frac{5}{9}(F - 32) \\ C &= \frac{5}{9}(44.7 - 32) \\ &= \frac{5}{9}(12.7) \text{ Maintains precisions to tenths place} \\ &\approx 7.055555 \text{ Maintains 3 sigfigs} \\ &\approx 7.06 \text{ So, round to 3 sigfigs} \end{aligned}$$

In the subtraction step the 32 is exact so the precision is determined by solely 44.7 (tenths place). In the multipication step the 5/9 is exact so the precision is determined by the 12.7 (three significant digits). ▲

Checkpoint 1.2.18 Calculate the following. Round using significant digits.
$845 \times 24.44 =$ _____
$60.96 \div 59 =$ _____

The rounding for significant digit rules is applied at the end of a calculation. That is if we have a mix of addition, subtraction, multiplication, and division then we do all of the operations, track the significant digits that should apply for each operation and apply the rounding at the end.

This example also illustrates tracking significant digits and illustrates a way you might keep track of which digits are significant in each step. The rightmost (smallest place value) digit that is significant is underlined. This is not a mathematical notation, but rather a convenient way to track the significant digits.

Example 1.2.19 Multi-Step Arithmetic with Significant Digits. Evaluate $11,728 + 39(17.9 + 1.23)$.

$$\begin{aligned} 11,72\underline{8} + 3\underline{9}(17.\underline{9} + 1.2\underline{3}) &= \\ 11,72\underline{8} + 3\underline{9}(19.\underline{1}3) &= \\ 11,72\underline{8} + 7\underline{4}6.07 &= 12,4\underline{7}4.07 \\ &\approx 12,470. \end{aligned}$$

By order of operations we first calculate $17.9 + 1.23 = 19.13$. Note that the result 19.13 is significant only to the first decimal place because 17.9 is only significant to the first decimal place. Second by order of operations we calculate $39 \cdot 19.13 = 746.07$. This is the product of a number with 2 significant digits and one with 3 significant digits so the result should have only 2 significant digits which would be the 10's place (the 4). The last calculation is $11,728 + 746.07 = 12,474.07$. 11,728 is significant to the one's place but the 746.07 is significant only to the 10's place. This means the final result is rounded to the 10's place so $11,728 + 39(17.9 + 1.23) \approx 12,470$. ▲

Example 1.2.20 Calculate

$$21 \cdot 9 - \frac{1}{2} \cdot 9(21 - 5 - 4).$$

The 1/2 is an exact number here. The rest are measurements in centimeters.

$$\begin{aligned} 2\underline{1} \cdot \underline{9} - \frac{1}{2} \cdot \underline{9}(2\underline{1} - \underline{5} - \underline{4}) &= \\ 2\underline{1} \cdot \underline{9} - \frac{1}{2} \cdot \underline{9}(1\underline{2}) &= && \text{Subtraction maintains precision to units} \\ \underline{1}89 - \underline{5}4 &= && \text{Multiplication maintains only 1 sigfig} \\ \underline{1}35 &\approx 100 && \text{Subtraction maintains only to hundreds} \end{aligned}$$

When we round we lose 35 units. If this seems like a lot consider the following. If all these measurements were rounded up (e.g., 21 was rounded from 20.5), then the value could have been

$$20.5 \cdot 8.5 - \frac{1}{2}8.5(20.5 - 4.5 - 3.5) = 121.125.$$

If all the measurements were rounded down (e.g., 21 was rounded from 21.4), the the value could have been

$$21.4 \cdot 9.4 - \frac{1}{2}9.4(21.4 - 5.4 - 4.4) = 146.64.$$

Using these two possibilities we see that the value could be anywhere from 121.125 to 146.64. The one hundred is the only digit about which we can be certain. That is why it is the only significant digit and why we eliminate (replace with zero) all the others.

If losing this much precision is a problem, then we need to obtain more precise measurements. ▲



Standalone
In Context

Significant digts communicates which digits are definitely meaningful. There are other motivations for rounding which lead to keeping more digits. This will be addressed in the next section.

Checkpoint 1.2.21 $(31.6 \cdot 3.3) + (61.2 \cdot 8.3) =$ _____

1.2.3 Rounding

Significant digits uses rounding to remove non-useful precision. This section presents various motivations for rounding and types of rounding and motivations for each.

Table 1.2.22 Reasons for Rounding

Reality Constraints	For example we cannot buy partial packages or have fractional people
Remove Detail	For example when describing the population of a nation
Control Error	When used in significant digits

The reason for rounding determines how we do it. Consider the following reality constraints requiring rounding. For example if we need 21 eggs and eggs are sold in cartons of one dozen (12) eggs, we need $21/12 = 1.75$ cartons. Since we cannot purchase part of a carton, we must round 1.75 up to 2, and purchase 2 cartons.

Note in this example reality requires us to round up to the nearest integer. We round to an integer because we cannot purchase fractional cartons of eggs. We had to round up, because rounding down would leave us with insufficient eggs (and we are hungry).

Suppose you have a bank account containing \$11,410 that accrues 1.65% interest. The bank calculates the payment should be $\$11,410 \cdot 0.0165 = \$188.265$. The bank will pay you \$188.26. They round to the nearest one hundredth because cents is a unit which can be paid. They round down, because they like paying less.

For removal of detail consider reporting the population of a country. We might report the population as over 9 million rather than 9,904,607. There are multiple motivations for this rounding. Note the population is likely changing multiple times per day, so more precision in the number does not equal more accuracy. Also, because of the scale (millions) the detail about how many ones, tens, hundred, and thousands looses meaning.

When reporting on salary ranges we might report a range between \$60,000 and \$80,000. That the range is actually \$61,233.57 to \$80,290.11 is unlikely to change a decision. The applicant will ask about the exact salary after deciding the position is a good fit. A common usage of removing detail is when we care about the scale of things rather than the count.

Rounding to control error is the use of significant digits.

Before considering context, we will practice rounding numbers. Note we can round to any digit. We can round up, down, or to the nearest number (what is meant by "round" if neither up nor down are specified). Context or instructions will specify which digit and which type of rounding.

Example 1.2.23 Rounding Up/Down.

(a) Round 72481 down to the nearest hundred.

Solution. 72400 is rounding down: we leave the 4 (hundred position) alone and "truncate" (turn to 0) all digits to the right. Note $72400 \leq 72481$.

(b) Round 72481 up to the nearest hundred.

Solution. 72500 is rounding up: we increase the 4 to a 5 and "truncate" (turn to 0) all digits to the right. Note $72500 \geq 72481$.

(c) Round 72481 the nearest hundred.

Solution. Because 72481 is closer to 72500 than it is to 72400, we round to 72500. We can recognize that we should round up because the tens position is $8 \geq 5$ which means rounding up results in a closer number. We could also recognize the need to round up by calculating $500-481 = 19$ and $481-400 = 81$ and noticing that $19 \leq 81$ (round up is closer).

▲

Example 1.2.24 Rounding Up/Down.

(a) Round 72481 down to the nearest thousand.

Solution. 72000 is rounding down: we leave the 2 (thousands position) alone and "truncate" (turn to 0) all digits to the right. Note $72000 \leq 72481$.

(b) Round 72481 up to the nearest thousand.

Solution. 73000 is rounding up: we increase the 2 to a 3 and "truncate" (turn to 0) all digits to the right. Note $73000 \geq 72481$.

(c) Round 72481 to the nearest thousand.

Solution. Because 72481 is closer to 72000 than it is to 73000, we round to 72000. We can recognize that we should round up because the hundreds position is $4 < 5$ which makes it closer to go down. We could also recognize the need to round down by calculating $3000 - 2481 = 519$ and $2481 - 2000 = 481$ and noticing that $481 < 519$.

▲

Example 1.2.25 Rounding to Different Precisions. Round 72321.83 to the specified precision.

- Thousands: 72000
- Ones:72322
- Tenths: 72321.8

▲

Checkpoint 1.2.26 Round 268479 as indicated below.
To the 10's position: _____
To the 100's position: _____
Up in the 100's position: _____
Down in the 100's position: _____

Next we need to consider when to use each type of rounding.

Example 1.2.27 Some floors are covered in carpet tiles. These are squares of carpet that are tiled to cover a floor. Suppose the carpet tiles are square with side length 20''. If a room is 50 feet by 38 feet, how many carpet squares do we need?

First lets figure out how many tiles will go across the 50 feet. Before we can we need to convert feet to

inches, so we can compare the width of the room to the carpet square width. 50 feet is 50 ft $\times \frac{12 \text{ in}}{\text{ft}} = 600$ in. We want to know how many 20 in tiles fit into 600 in. $\frac{600 \text{ in}}{20 \text{ in}} = 30$ tiles across. Notice that the units divide out which indicates we set this up correctly.

Next, lets figure out how many tiles will go across the 38 feet. 38 feet is 38 ft $\times \frac{12 \text{ in}}{\text{ft}} = 456$ in. This will require laying $\frac{456 \text{ in}}{20 \text{ in}} = 22.8$ tiles across. For each 0.8 of a tile we must cut a tile leaving only 0.2 of a tile left. This is too small to use elsewhere. Thus for each of these we will use a whole tile resulting in needing 23 tiles across (rounding up to have enough).

Finally we can count the number of tiles which is $30 \times 23 = 690$ tiles.

Could we use the 0.2 part of a tile that results from cutting at the end in another row. That depends. Doing so will start the next row shifted by 0.2, so if the tiles need to be aligned due to their color/pattern this will not work. If the color is solid or designed to be offset, we can do this and save money. Knowledge from carpet installation is required to determine this rounding. ▲

Example 1.2.28 Suppose you baked three (3) dozen cookies and are distributing them equally between 7 people. How many cookies does each person receive?

There are $3 \cdot 12 = 36$ cookies. Each person can have $36/7 \approx 5.1$ cookies. Because cutting cookies into pieces is how the cookie crumbles, we must round this down to 5 cookies per person.

Curious minds want to know what happens with the rest of the cookies. Notice there will be $7 \cdot 5 = 35$ cookies given away leaving just one cookie which the baker can enjoy. ▲

1.2.4 Greatest Possible Error

We have acknowledged that measurements will always have error. We have considered ways to round that are practical for the circumstances. Part of this depends on controlling the error. This section presents how to calculate the maximum error (worst case scenario). Typically we use this to ensure that error will not cause problems.

Because our rule for rounding is digits 0-4 round down and digits 5-9 round up, this rounding will always result in a greatest possible error of 5 in the position to the right of the one rounded. Consider the following.

Example 1.2.29 What is the greatest possible error if 130 was rounded to the nearest 10?

One possibility is that 130 was rounded down. Then the original number was one of 130, 131, 132, 133, or 134. 134 is the farthest away from 130 at $134 - 130 = 4$.

The other possibility is that 130 was rounded up. Then the original number was one of 125, 126, 127, 128, or 129. 125 is the farthest away at $130 - 125 = 5$.

Thus the greatest possible error was 5 from the case that 125 was rounded up.

Note in this solution we assumed the number rounded was an integer. However, if we allowed for 134.927 and 125.01 the result would be the same. the extra digits do not change the rounding. ▲

Example 1.2.30 What is the greatest possible error if 9.31 was rounded to the nearest hundredth?

Solution. The largest possible error is if 9.31 was rounded up from 9.305. Thus the greatest possible error is 0.005 or 5 one thousandths. ▲

To determine the greatest possible error, we have to know what type of rounding was used. In most measurements (i.e., significant digits) standard rounding will be used. For example think about measuring on a ruler: if the object is not exactly on one of the lines, you will choose the closest one (left or right). Moving to the closest one is rounding.

The next example calculates the greatest possible error from rounding up (as opposed to rounding).

Example 1.2.31 What is the greatest possible error if 223 was rounded up to the nearest unit?

Solution. 223 could have been rounded up from 222.1. But it could also have been rounded up from 222.01 or anything else. Thus the greatest possible error is less than 1 ($223 - 222 = 1$). ▲

Checkpoint 1.2.32 What is the greatest possible error of 29800 if it was rounded to the nearest 100? ___

Checkpoint 1.2.33 What is the greatest possible error of $131,\overline{0}00$? ___

1.2.5 Exercises

Counting Significant Digits. Determine the number of significant digits using the standard notations.

1. How many significant figures does 303,000,000,000 s have?

_____

2. Tell how many significant digits there are in each measurement.

1. 23,000.0 km _____
2. 2041 m _____
3. 0.03 gal _____
4. 423,000,000 ft _____

3. How many significant digits does 3.980 270 0 have?

1
2
3
4
5
6
7
8

4. Determine how many significant figures are in each measurement.

1. 1199.011 mL

   (a) 1
   (b) 2
   (c) 3
   (d) 4
   (e) 5
   (f) 6
   (g) 7
   (h) 8
   (i) 9
   (j) 10

2. 87 in

   (a) 1
   (b) 2
   (c) 3
   (d) 4
   (e) 5
   (f) 6
   (g) 7
   (h) 8
   (i) 9

(j) 10

3. 700 m

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5
(f) 6
(g) 7
(h) 8
(i) 9
(j) 10

4. 1475.56 g

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5
(f) 6
(g) 7
(h) 8
(i) 9
(j) 10

5. 700.0 m

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5
(f) 6
(g) 7
(h) 8
(i) 9
(j) 10

5. Determine the accuracy (i.e., the number of significant digits) of this number: 0.0106

123
4
5

6. Determine the accuracy (i.e., the number of significant digits) of this number: $454,400$

1
3
4
5
6

Significant Digit Arithmetic. Perform the arithmetic indicated and write the result with the correct number of significant digits.

7. a) Round 855.4363 g to 2 significant digits.

_____

b) Round 855.4363 g to 4 significant digits.

_____

c) Round 855.4363 g to 3 significant digits.

_____

8. Round off the approximate number as indicated.

11.68; 2 significant digits

_____

9. Calculate the product below, and express the result with the correct number of significant figures.

$4.6 \times 6.349 =$ _____

10. Calculate the quotient below, and express the result with the correct number of significant figures.

15.1 ÷ 3.887 = _____

11. Calculate the following and round using sigfigs. The 2 is an exact number.

$\frac{2.25}{2}(53.68 + 13.47) =$ _____

12. Calculate the following and round using sigfigs.

$6.8 \cdot 4.2 + 5.6 \cdot 1.1 =$ _____

13. Calculate the following and round using sigfigs.

$(3.14159)\Big[(13)^2(44) + (9.0)^2(40.)\Big] =$ _____ cm^2

14. Calculate the following and round using sigfigs.

$(12.4 - 9.5)(12.4 - 6.4)(12.4 - 8.8) =$ _____

15. Calculate the following and round using sigfigs.

$\frac{\pi}{3}\big(1.10^2 \cdot 5.00 - 0.836^2 \cdot 3.80\big) =$ ____

16. Torque is used to measure how tight a bolt is. The unit is foot-pounds (ft-lbs). For example 5 ft-lbs is the tightness obtained by putting 5 lbs (weight) at the end of a one foot long wrench.

Guido decides to experiment with torque by hanging from a wrench. If Guido weighs 68 lbs and hangs on the wrench 14 inches from the center of the bolt, what torque is generated? _____

Because this is an experiment, round according to the rules for measurement.

Guido realizes that his hand is 2.5 inches wide, and he might not be putting his weight in the exact center of his hand on the wrench. If the center of his hand is 14 inches from the center of the bolt, what is the expected torque if his weight is all on the side of his hand closest to the bolt?

_____

What if his weight is all on the side of his hand farthest from the bolt? _____

If the torque produced from the center of his hand is the correct one, what is the greatest possible error? (Use your calculations above) _____

Greatest Possible Error. Determine the greatest possible error for each number or calculation.

17. Determine the greatest possible error of: 0.208

1. ± 0.5

2. ±0.05
3. ±0.005
4. ±0.0005
5. ±0.00005

18. Given the measurement 4.1 gal, find the following.
Precision to nearest _____ (thousand, hundred, ten, whole, tenth, hundreth, thousandth)
Accuracy _____ (number of significant digits)
Greatest possible error _____ gal

19. Given the measurement 0.009 ft, find the following.
Precision to nearest _____ (thousand, hundred, ten, whole, tenth, hundreth, thousandth)
Accuracy _____ (number of significant digits)
Greatest possible error _____ ft

20. Given the measurement 367 psi, find the following.
Precision to nearest

1. thousand
2. hundred
3. ten
4. whole
5. tenth
6. hundreth
7. thousandth

Accuracy _____ (number of significant digits)
Greatest possible error _____ psi

1.3 Working with Applications

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Solve *linear*, rational, quadratic, and exponential equations and formulas (skill)
- Read and interpret models (critical thinking)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

In life when we figure out processes at work or in science we often express the result in mathematical notation. This includes equations, functions, and other options. These are collectively known as models. They allow us to communicate what we know and calculate results as needed. To succeed in many jobs and to fully enjoy life we need to be proficient at reading and using models.

This section begins by presenting models and illustrates calculating some results from them. It progresses to solving equations (models) as a review of algebra skills. Finally we present tips on how to identify and use models arising in applications. These topics are continued with specific models in later sections.

1.3.1 Calculating Results using Models

Model 1.3.1 Ohm's Law. *Ohm's Law relates three properties of electricity: voltage, current, and resistance.* ***Voltage****, measured in volts (V), is analagous to the amount of pressure to move the electrons.* ***Current****, measured in amperes (amps), is how much electricity is moving.* ***Resistance*** *measured in Ohms (Ω), is, as it sounds, the resistance of a material to letting electricity flow.*

The relationship is

$$V = IR$$

where

- *V is voltage,*
- *I is current, and*
- *R is the resistance.*

Example 1.3.2 We will consider how voltage must change as the needed current changes. These are measurements in a science model which means we will use significant digits rounding.

Suppose the current is 3.0 amps and the resistance is 8.0 ohms. What is the voltage?

Using the model we can calculate

$$V = 3.0 \cdot 8.0 = 24.$$

Thus in this system 24 volts is required.

If the current is increased to 6.0 amps on the 8.0 ohm circuit, then

$$V = 6.0 \cdot 8.0 = 48.$$

This result tells us that to double the amps, doubling the voltage is required.

Similarly if we know that the current is 1.7 amps and the resistance is 6.0 ohms, then we can calculate

$$V = 1.7 \cdot 6.0 = 10.2 \approx 10.$$

Thus in this system 10 volts is required. ▲

Model 1.3.3 Lift Equation. *Lift is the force that keeps aircraft in the air.The lift equation explains factors that control the strength of lift produced by an airfoil (think wing or propellor). The factors included are air density, surface area of the airfoil, the coefficient of lift, and velocity. Air density is the amount of air per volume; you may see this as highs and lows on a weather map. It is also related to pressurizing aircraft flying at high altitude. The coefficient of lift incorporates multiple factors that are part of the design of the airfoil and how it is in use during flight.*

The lift equation is

$$L = \frac{1}{2}\rho s C_L v^2$$

where

- *L is the lift in units of lbs or Newtons)*
- *ρ is air density in units of slugs per cubic feet or kilograms per cubic meter*
- *s surface area in units of square feet or square meters*
- *C_L is the coefficient of lift which is unitless*
- *v is velocity in units of feet per second or meters per second*

Example 1.3.4 We can consider the impact of changes in air density. Air density generally decreases as altitude increases. This is a science model with measurements, so we will use significant digits rounding.

Suppose the air density is 0.002378 slugs per cubic feet, surface area is $125\,\text{ft}^2$, $C_L = 1.5617$, and velocity is $84.4\,\frac{\text{ft}}{\text{s}}$. Calculate the lift.

$$\begin{aligned} L &= \frac{1}{2} \cdot 0.002378 \cdot 125 \cdot 1.5617 \cdot 84.4^2 \\ &\approx 1653.386439 \\ &\approx 1650. \end{aligned}$$

Under these circumstances this airfoil can lift 1650 lbs.

If the air density is reduced to 0.001988 slugs per cubic feet, what is the lift? This represents the same aircraft flying 6000 feet higher.

$$\begin{aligned} L &= \frac{1}{2} \cdot 0.001988 \cdot 125 \cdot 1.5617 \cdot 84.4^2 \\ &1382.225501 \\ &\approx 1380. \end{aligned}$$

If nothing else is changed (like velocity), this airfoil can lift (hold in the air) $1650 - 1380 = 270$ lbs less. ▲

Model 1.3.5 Ideal Gas Law. *The ideal gas law is a relationship between the volume, pressure, temperature, and number of molecules of an ideal gas. The relationship is*

$$PV = nRT$$

where

- *P is the pressure in units of atmospheres (atm) or Pascals (Pa)*
- *V is the volume in units of cubic feet or cubic meters*
- *n is the number of moles (number of molecules, see a chemistry text for details)*
- *R is a constant specific to each gas (e.g., oxygen and nitrogen have different ones) in units that match the other values*
- *T is the temperature in degrees Rankine or Kelvin (these are shifted versions of Fahrenheit and Celsius).*

When the number of molecules remains fixed, such as in a closed container, this law can be used to produce the equation

$$\frac{P_1 V_1}{T_1 + 273} = \frac{P_2 V_2}{T_2 + 273}.$$

where

- *P_1, V_1, and T_1 are the initial pressure, volume, and temperature, and*
- *P_2, V_2, and T_2 are the pressure, volume, and temperature at another time.*

Note in both forms of the law the units can be other than those listed (especially different scale like centimeters rather than meters). However, they must always match including the constant R which is looked up in reference books.

For rounding note 273 is three (3) significant digits. 273.15 can be used if more precision is needed.

Example 1.3.6 Suppose the initial conditions are $P_1 = 101.3$ Pa, $V_1 = 0.125$ m^3, and $T_1 = 10.2°$ C. Also $V_2 = 0.125$ m^3 and $T_2 = 50.7°$ C. Calculate the new pressure. This is a science model with measurements,

so we will use significant digits rounding.

$$\begin{aligned}
\frac{101.3 \cdot 0.125}{10.2+273} &= \frac{P_2 \cdot 0.125}{50.7+273} && \text{Addition maintains precision to tenths}\\
\frac{101.3 \cdot 0.125}{10.2+273} &= \frac{P_2 \cdot 0.125}{323.\underline{7}} && \text{Multiplication maintains 3 sigfigs}\\
\frac{12.\underline{6}625}{283.\underline{2}} &= \frac{P_2 \cdot 0.125}{323.\underline{7}} && \text{Division maintains 3 sigfigs}\\
0.044\underline{7}122 &\approx 0.00038\underline{6}160P_2 && \text{Divide to isolate } P_2\\
\frac{0.044\underline{7}122}{0.00038\underline{6}160} &\approx \frac{0.00038\underline{6}160P_2}{0.00038\underline{6}160} && \text{Division maintains 3 sigfigs}\\
11\underline{5}.786721 &\approx P_2\\
116 &\approx P_2.
\end{aligned}$$

If instead $T_2 = -10.3°$ C, then we have the following.

$$\begin{aligned}
\frac{101.3 \cdot 0.125}{10.2+273} &= \frac{P_2 \cdot 0.125}{-10.3+273}\\
\frac{12.\underline{6}625}{283.\underline{2}} &= \frac{P_2 \cdot 0.125}{262.\underline{7}}\\
0.044\underline{7}122 &\approx 0.0004\underline{7}5828P_2\\
\frac{0.044\underline{7}122}{0.0004\underline{7}5828} &\approx \frac{0.0004\underline{7}5828P_2}{0.0004\underline{7}5828}\\
93.\underline{9}6714779 &\approx P_2\\
94.0 &\approx P_2
\end{aligned}$$

▲

Checkpoint 1.3.7 Recall Ohm's Law states $V = IR$.
Calculate the voltage if the current is $I = 4.8$ and the resistance is $R = 7$. $V =$ ____

Checkpoint 1.3.8 Note $\frac{V_1P_1}{T_1+273} = \frac{V_2P_2}{T_2+273}$ with temperature in Celsius.
If $V_1 = V_2 = 0.250$, $P_1 = 2381$, $T_1 = 19.0$, and $T_2 = 30.0$, what is P_2? _____

1.3.2 Calculating Results Requiring Solving

The previous section illustrated calculating model results without solving. This section presents additional example requiring limited solving and finishes with solving before any values have been substituted.

It does not matter if the value we desire is by itself, we can solve using arithmetic.

Example 1.3.9 Recall the model for lift. Suppose we know the weight of the aircraft ($w = 239\bar{0}$ lbs), density of air ($\rho = 0.001869$ slugs/ft^3), wing surface area ($s = 165$ ft^2), and velocity ($v = 91.1$ ft/sec). Noting that lift must equal weight, what must the coefficient of lift be? This is a science model with measurements, so we will use significant digits rounding.

$$\begin{aligned}
L &= \frac{1}{2}\rho s C_L v^2\\
239\bar{0} &= \frac{1}{2}(0.001869)(165)C_L(91.1)^2. \text{ Multiply.}\\
239\bar{0} &\approx 12\underline{7}9.675938C_L. \text{ Multiplication maintains 3 sigfigs}\\
\frac{239\bar{0}}{12\underline{7}9.675938} &\approx \frac{12\underline{7}9.675938C_L}{12\underline{7}9.675938} \text{ Divide to isolate } C_L.
\end{aligned}$$

$$1.8\underline{6}7660342 \approx C_L. \text{ Division maintains 3 sigfigs}$$
$$1.87 \approx C_L.$$

▲

The desired value from the model may be in a denominator. We can solve for this using multiplication and division.

Example 1.3.10 Recall from Model 1.3.5 that under simplifying assumptions

$$\frac{P_1V_1}{T_1+273} = \frac{P_2V_2}{T_2+273}.$$

Suppose we know the initial conditions ($P_1 = 1.00$ atm, $V_1 = 1.35$ ft^3, $T_1 = 51.2°$ F) and also $P_2 = 1.00$ atm and $V_2 = 1.39$ ft^3. What must the new temperature (T_2) be? This is a science model with measurements, so we will use significant digits rounding.

$$\frac{1.00 \cdot 1.35}{51.2+273} = \frac{1.00 \cdot 1.39}{T_2+273}. \text{ Perform arithmetic}$$
$$\frac{1.3\underline{5}}{324.\underline{2}} = \frac{1.3\underline{9}}{T_2+273}. \text{ More arithmetic}$$
$$0.0041\underline{6}40962 \approx \frac{1.3\underline{9}}{T_2+273}. \text{ Multiply to move } T_2 \text{ out of denominator.}$$
$$0.0041\underline{6}40962(T_2+273) \approx \frac{1.3\underline{9}}{T_2+273}(T_2+273).$$
$$0.0041\underline{6}40962(T_2+273) \approx 1.3\underline{9}. \text{ Divide to undo multiplication.}$$
$$\frac{0.0041\underline{6}40962(T_2+273)}{0.0041\underline{6}40962} \approx \frac{1.3\underline{9}}{0.0041\underline{6}40962}. \text{ Division maintains 3 sigfigs}$$
$$T_2+273 \approx 33\underline{3}.80593. \text{ Subtract to undo addition.}$$
$$T_2+273-273 \approx 33\underline{3}.8059259 - 273.$$
$$T_2 \approx 6\underline{0}.80593. \text{ Subtraction maintains precision to units}$$
$$T_2 \approx 61.$$

Note we use significant digits for rounding because this is a science model. ▲

The previous examples solved for a variable in a model after substituting numbers for the other variables. The next examples illustrate solving first. Note this process is the same as solving after substituting (same algebra) though there may be more steps. We might wish to solve this way, so it is easier to use the model multiple times.

Example 1.3.11 Solve the equation $V = IR$ for R. Note, this model is explained in Model 1.3.1.

$$V = IR.$$
$$\frac{V}{I} = \frac{IR}{I} \quad \text{Divide to undo multiplication.}$$
$$\frac{V}{I} = R.$$

▲

Example 1.3.12 Solve the lift equation $L = \frac{1}{2}\rho S C_L v^2$ for S.

$$L = \frac{1}{2}\rho S C_L v^2.$$

$$
\begin{aligned}
2L &= 2\frac{1}{2}\rho S C_L v^2. && \text{Multiply to undo division.}\\
2L &= \rho S C_L v^2.\\
\frac{2L}{\rho} &= \frac{\rho S C_L v^2}{\rho}. && \text{Divide to undo multiplication.}\\
\frac{2L}{\rho} &= S C_L v^2.\\
\frac{2L}{\rho C_L} &= \frac{S C_L v^2}{C_L}.\\
\frac{2L}{\rho C_L} &= S v^2.\\
\frac{2L}{\rho C_L v^2} &= \frac{S v^2}{v^2}.\\
\frac{2L}{\rho C_L v^2} &= S.
\end{aligned}
$$

Notice that the steps are the same algebra as if there were numbers. Also we could divide by v^2 and we are not concerned with the square as part of solving for S. ▲

Checkpoint 1.3.13 Solve the lift equation $L = \frac{1}{2}\rho S C_L v^2$ for ρ.

1.3.3 Process Overview

Above we started with a model and were asked to do something with it. Normally we will start with a problem which does not identify a model to use. Very few problems you encounter in lifte come pre-labled with models. This section presents how to start with a problem and work to a solution by identifying the model first.

Our first task is to read the problem to understand it.

- Read the problem description a few times.
  - If you can paraphrase it, you understand it enough.
  - Drawing a picture and labeling parts may help.
- Identify what we are asked to do.
- Identify the information we are given. Note distinctions like measurements and rates.
- Identify any units. These often help us set up a model.
- Write everything! We do not model in our heads.

Next we write the mathematical model (equation or function).

- Use the description to determine which application type (e.g., percent, proportion, linear model, etc.). Note units can suggest this (e.g., meters and meters squared indicate something was squared).
- Do not insert any numbers yet.
- Do not do any calculations yet.

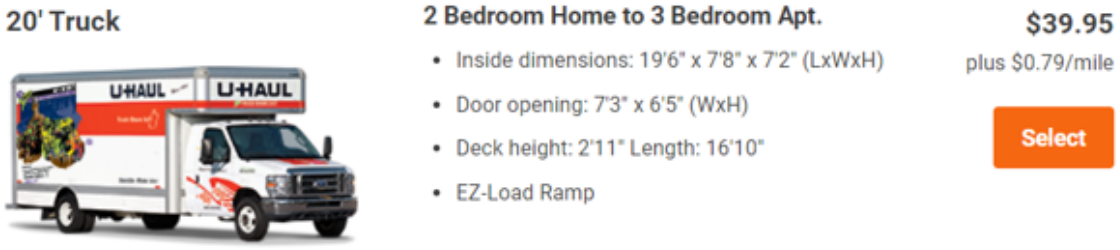
Now we will have a model that matches our situation and possibly some numbers to insert.

- Insert numbers into the model. You may have to calculate some of these (e.g., you are given two points but not the slope you need).
- Solve for the desired value. Note it may help to do some calculations with the numbers first.

- State your answer and use units appropriately.

Finally we should check that our answer makes sense. We should not have negative prices (usually) or distances larger than the earth (when working with terrestrial problems).

Example 1.3.14 You moved across town and rented a 20 foot moving truck for the day. You want to make sure the bill you received is correct. If you paid \$81.03, for how many miles were you charged? Assume there were no extra fees.



Solution. We want to compare the bill we received to the price listed in the add. The question is about how many miles (not how much money).

We are given the price per mile (\$0.79 per mile). There is also a fixed cost for the rental (\$39.95). Adding the fixed cost and the milage cost will give us the total.

Our model is $C = \$39.95 + \$0.79m$ where C is the total cost and m is the number of miles.

We know the total cost, which will leaves m in the equation, the number of miles, which is what we want to calculate. We can use the solving technique in Section 3.1

$$\begin{aligned} \$81.03 &= \$39.95 + \$0.79m \\ -\$39.95 + \$81.03 &= -\$39.95 + \$39.95 + \$0.79m \\ \$41.08 &= \$0.79m \\ \frac{\$41.08}{\$0.79} &= \frac{\$0.79}{\$0.79}m \\ 52 &= m. \end{aligned}$$

The charge is for 52 miles. ▲



Standalone
In Context

Figure 1.3.15 Using math modeling for rental truck

1.3.4 Exercises

1. The *delivery* of a pump is the amount of fluid supplied per unit of time. This can be calculated as follows.

$Q = \dfrac{D \times R}{231}$

where

Q is pump output flow

D is pump displacement (in units of cubic inches per revolution)

R is rotations per minute (RPM)

231 is for unit conversion (in units of cubic inches per gallon)

If a pump displaces $D = 0.91$ in^3/revolution and is operating at $R = 1034$ rotations per minute, what is the output flow of this pump? _____

2. The heat gain or loss due to change of pressure in a hydraulic circuit can be calculated using

$\Delta h = \Delta P \times Q \times 1.5$

where

Δh is the change in temperature in BTUs per hour

ΔP is the change in pressure in PSI

Q is the flow through the circuit in GPM

If the pressure changes from 3857 psi to 4137 psi, and the flow is 23 GPM, what is the temperature change? _____ BTU/hr

3. How deeply our feet press into snow or sand depends on the pressure. Pressure is given by

$P = \dfrac{F}{A}$ where

P is the pressure

F is the force in lbs, and

A is the area in square inches

Suppose Aniyah weighs 131 lbs and each foot has area 72 square inches. While standing still what pressure does Aniyah exert on the ground? ____ psi

4. In a thunderstorm, the formula:

$M = \dfrac{t}{5.4}$

gives the approximate distance, M, in miles, from a lightning strike if it takes t seconds to hear the thunder after seeing the lightning. If you are 6.2 miles away from the lightning flash, how long will it take the sound of the thunder to reach you.

Answer: It will take _____ seconds for the sound to reach you.

5. You and your classmates create 22 abstract paintings to sell at the PTA auction to raise money for your school. In order to pay for the materials you bought to make the abstract paintings, you need to sell each of the abstract paintings for $5. When transporting the abstract paintings to the auction, one of your parents accidentally drops four of the abstract paintings in the street and they are run over by a PeopleMover Transit bus. What is the new minimum price you need to set for the remaining abstract paintings in order to pay for the materials?

Keep in mind that you do not want to collect less money than you paid for the materials. This may affect how you round your answer. $____

6. A bookshelf containing 7 bookshelves is to be constructed. The floor-to-ceiling clearance is 9 ft 6.0 in. Each shelf is 1.0 in thick. An equal space is to be left between the shelves, the top shelf and the ceiling, and the bottom shelf and the floor. (There is no shelf on the ceiling or floor.)

What space should be between each shelf and the next? Round your answer to the nearest tenth of an inch.

_____ in

7. A student in a Pre-Calculus class has test scores of 77, 72, 73, and 61. The final exam is worth 3 test grades.

Write a linear equation that models this problem, where x is the grade on the final exam and y is

the student's grade in the course. The whole grade is based on these tests and the final.

_____

What grade is needed on the final to earn a B (average score of 80%)?

_____

8. Amanda rented a bike from Ted's Bikes.

It costs \$11 for the helmet plus \$3.75 per hour.

If Amanda paid about \$24.13, how many hours did she rent the bike?

a) Let h = the number of hours she rented the bike. Write the equation you would use to solve this problem.

_____

b) Now solve your equation

Amanda rented the bike for _____ hours. (Round your answer to the nearest tenth of an hour.)

9. Pilot training requires ground school and flight time. Suppose ground school costs \$702 and flight training costs an average of \$323 per hour.

Write an expression for the total cost of training with t as the variable for time. _____

If it takes 61 hours, what is the total cost of training? _____

10. The area of a triangle is given by the formula $A = \frac{1}{2}bh$, where A is the area of the triangle, b is its base and h is its height.

Solve the formula, $A = \frac{1}{2}bh$, for h.

$h =$ _____

Find the height of the triangle with base of 7 meters and area of 150 square meters. Write your answer as a decimal, rounded to the nearest hundredth, when necessary.

height = _____ meters

11. It takes Rylla 18 hours to knit a scarf. She can only knit for 1.5 hours per day. How many days will it take her to knit the scarf?

Part 1: Let x be the number of days it will take her to knit the scarf. Choose the correct translation of this problem into an equation:

1. x=(1.5)(18)
2. 18-1.5=x
3. 1.5 x=18

Part 2: Solve for x.

_____

12. A rental car company charges \$50 plus 40 cents per each mile driven.

Part1. Which of the following could be used to model the total cost of the rental where m represents the miles driven.

1. $C = 4m + 50$
2. $C = 40m + 50$
3. $C = 0.4m + 50$

Part 2. The total cost of driving 150 miles is;

\$_____

1.4 Project: Literal Formula

Project 1.4.1 Literal Formula. Most math books define the area of a circle as follows: $A = \pi r^2$, where A is the area of the circle and r is the radius of a circle. A text used in UAA's Automotive Diesel program defines the area of a circle as $A = 0.7854d^2$, where A is the area of the circle and d is the diameter of the circle.

The purpose of this project is to determine when each formula is most useful.

(a) What is the mathematical relationship between radius and diameter? Your answer can be a sentence or an equation.

(b) Show mathematically how to get from the formula $A = \pi r^2$ to the formula $A = 0.7854d^2$. This should take you multiple steps.

(c) Explain in words what you did in each step to change the first formula into the second. What assumptions did you have to make? Anyone reading this answer should be able to replicate the math by just reading your answer. That is, talk me through all the steps.

(d) Did you have any false starts or did you see how to change the formula right away? There is no wrong answer here; I just want you to think about your process.

(e) For this problem, you will need a tape measure or a ruler. If doing this on a device it must be a computer and ensure you are at 100% magnification. Your phone or a scaled version will distort the results. First *measure* the radius of the circle in Figure 1.4.2. Then *measure* the diameter of the circle below and record your answer. ***Do not calculate the diameter!*** This must be measured, not calculated. Try to be as precise as is reasonably possible. Include units.

Was it easier to measure the radius or the diameter?

(f) What is one reason why it might be more practical on a job to use the formula $A = 0.7854d^2$ instead of $A = \pi r^2$? If it helps, you may wish to ask yourself why the auto diesel students in particular use this less traditional formula.

(g) Determine how many significant figures are in each measurement. If the number is not a measurement or the measurement has no error then it is called 'exact'.

- π
- 0.7854

(h) Which of the two formulas is more accurate? Which is more precise? Give a reason to back up your answer.

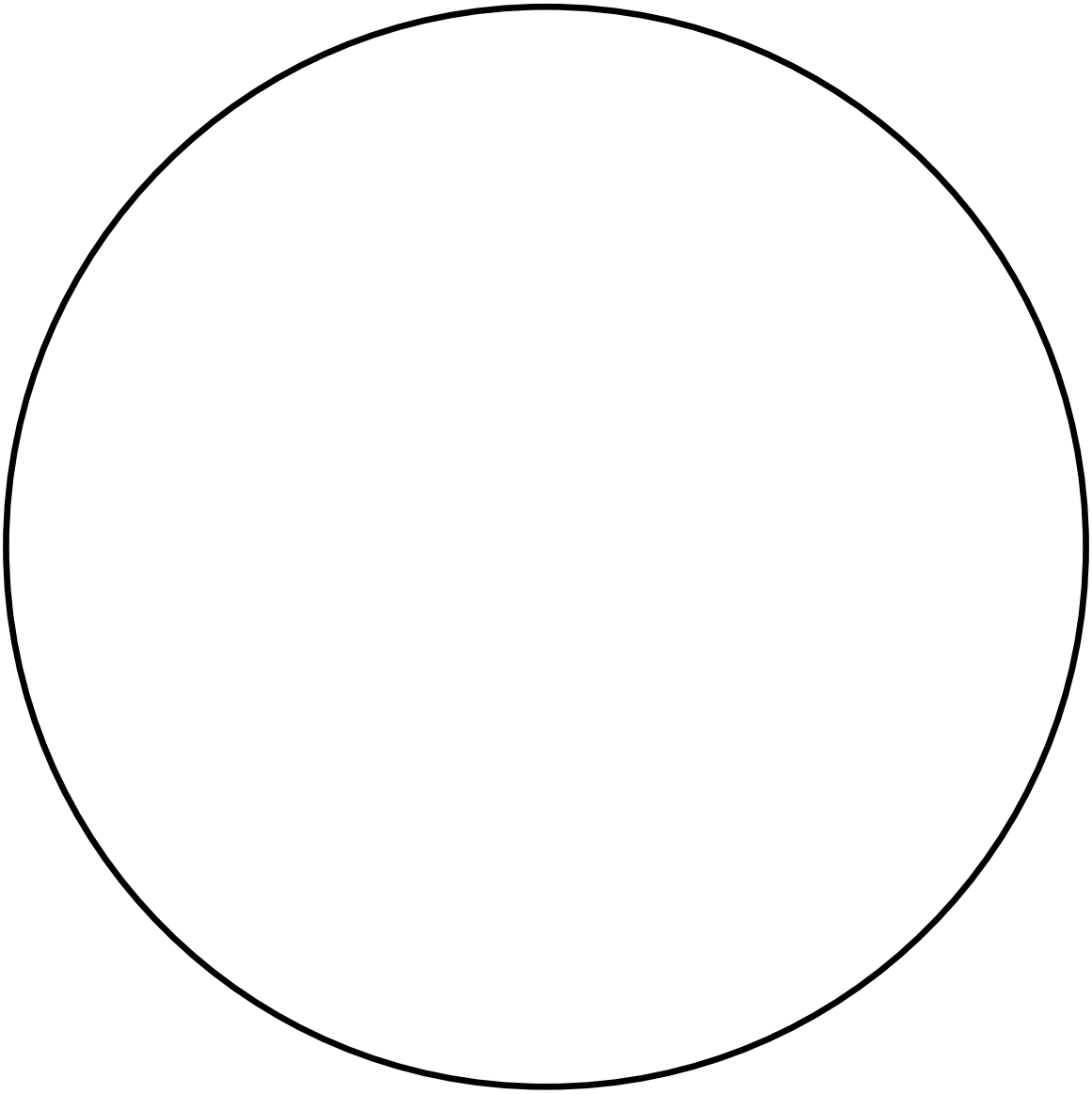


Figure 1.4.2 Circle

Chapter 2

Ratios

2.1 Percents

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Calculate Percentages (skill)
- Understand and interpret percentages (critical thinking)

Percentages are an often convenient way to express the relative size of two quantities such as the number of people who like lemon meringue pie to the number of those who like pie.

We will learn how to calculate a percent (Subsection 2.1.1, Item 2.b), how to convert between percents and the numbers (also Subsection 2.1.1, Item 2.b), how to describe growth in terms of percents (Subsection 2.1.2, Item 2.c), and how to recognize what a percent does and cannot tell us (Subsection 2.1.3, Item 2.c).

2.1.1 Calculating Percents

Definition 2.1.1 Percent. A **percent** is a ratio of part of something to the whole of that thing that is written as parts per hundred. ◆

Example 2.1.2 Calculate a Percent. In a class there are 34 students. Of them 21 are female. In this case female students is part of the whole (all students). Thus the percent is calculated as

$$\frac{\text{part}}{\text{whole}} = \frac{21}{34} \approx 0.6176.$$

This number says there are 61 hundreths (remembering our numbering system), so the percent is written as 61.76%.

Rounding to two (2) decimal places was chosen to illustrate how we convert a ratio in decimal form to a percent. If we were reporting this information, we would most likely round to 62%. This would convey the same meaning, because the difference between 61.76% and 62% for 34 people is less than one person. ▲



Standalone
In Context

Generally, we calculate a percent by

$$100 \times \frac{\text{part}}{\text{whole}}.$$

Example 2.1.3 In the class there are 34 students. Of them 13 are male. The percent is calculated as

$$100 \times \frac{13}{34} = 100 \times 0.3824 = 38.24\%.$$

▲

Now that we have presented two examples of calculating a percent from counts, use the check point below to test that you can setup and calculate one yourself.

Checkpoint 2.1.4 In another class there are 82 students and 49 are female. What percent of the students are female? _____

In the first pair of examples we had a whole class of 34 students with 21 female and 13 male. Of course 21+13 = 34, that is the two parts add up to the whole. Because of this 61.76% + 38.24% = 100% as well.

Sometimes we are given the size of the whole and a percent, and we are interested in calculating how many are in the part.

Example 2.1.5 In a class of 22 students, 18% are Alaska Native. How many students are Alaska Native?

We use the same setup as before, but we do not know the part yet.

$$\begin{aligned} 100 \cdot \frac{P}{22} &= 18\% \text{ Divide to isolate } P \\ \frac{P}{22} &= \frac{18}{100} \\ \frac{P}{22} &= 0.18 \text{ Multiply to isolate } P \\ P &= 22 \cdot 0.18 \\ &= 3.96. \end{aligned}$$

Notice that 3.96 does not make sense as a result when counting people, so we expect that the correct result is 4. We can confirm this by checking that

$$\begin{aligned} 100 \times \frac{\text{part}}{\text{whole}} &= 100 \times \frac{4}{22} \\ &= 100 \times 0.18181818 \\ &\approx 18.18\% \end{aligned}$$

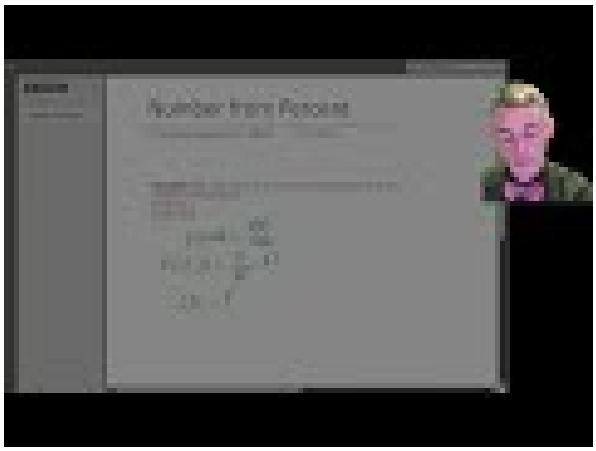
This suggests that the original 18% was rounded. Likely it was rounded to the ones position out of convenience.

We can solve this another way. We know that a percent is a number out of 100, so we can skip a step

from the previous example.

$$\frac{P}{22} = 0.18.$$
$$P = 22 \cdot 0.18.$$
$$P = 3.96.$$

▲



Standalone
In Context

Checkpoint 2.1.6 There are 37 students in a class. Below are percents for each racial group tracked. Calculate the number of students in each group.

Group	Percent	Number
Alaska Native	6.25%	___
Asian	12.5%	___
Black	6.25%	___
White	71.88%	___
Other	2.7%	___

Sometimes we know the size of a part and what percent it is of the whole. From this information we can calculate the size of the whole.

Example 2.1.7 In a class 2 Alaska Native students make up 6.25% of the class. How many students are in the class?

Again we use the same setup, but we do not yet know the whole.

$$\frac{\text{part}}{\text{whole}} = \text{percent}.$$
$$\frac{2}{W} = 0.0625.$$
$$\frac{2}{W} \cdot W = 0.0625 \cdot W. \text{ Move } W \text{ out of the denominator}$$
$$2 = 0.0625 \cdot W.$$
$$\frac{2}{0.0625} = \frac{0.0625 \cdot W}{0.0625}. \text{ Divide to isolate } W$$
$$\frac{2}{0.0625} = W.$$
$$32 = W.$$

▲

Example 2.1.8 How to Use an Example: Percents. Consider the following question.

If the first chapter of a certain book is 18 pages long and makes up 2% of the book, how many pages does the entire book have?

Because we see "2%" we recognize this as a percent problem. Without more information we can begin writing. In Example 2.1.7 the first step is writing the definition of percent.

$$\frac{\text{part}}{\text{whole}} = \text{percent}$$

In the example 0.0625 is written on the right (in place of percent). In this problem we know the percent is 2. We also know that in a calculation we convert the percent to a decimal. In this case $2\% = \frac{2}{100} = 0.02$. Thus the next step is

$$\frac{\text{part}}{\text{whole}} = 0.02$$

In the next step in the example the entries for part and whole are entered. In this problem the 18 pages is stated as one chapter and is contrasted to the "entire" book. Thus the 18 is the part. As with the example, the whole is not known so we leave it as a variable.

$$\frac{18}{W} = 0.02$$

Finally in the example they solve for the variable. Note the steps of solving may vary depending on what we know, so rather than follow the rest of the example step-by-step, we apply our algebra skills.

$$\begin{aligned}
\frac{18}{W} &= 0.02 \\
\frac{18}{W}W &= 0.02W \text{ Multiply to move } W \text{ out of denominator} \\
18 &= 0.02W \\
\frac{18}{0.02} &= \frac{0.02}{0.02}W \text{ Divide to isolate } W \\
900 &= W
\end{aligned}$$

Thus we know the entire book has 900 pages. ▲

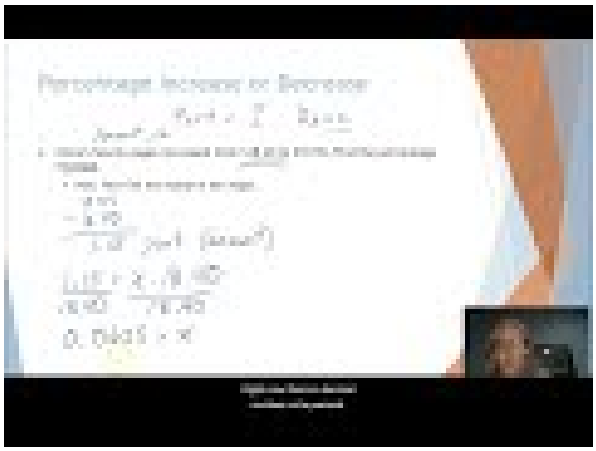


Standalone
In Context

In this next check point the terminology is different but something is still part of a whole and the amount can be calculated using the same approach as above.

Checkpoint 2.1.9 Find the number of millilitres of alcohol needed to prepare 120 mL of solution that is 7% alcohol. _____

This video covers the topics above.



Standalone
In Context

2.1.2 Percent Increase/Decrease

A common use of percents is to indicate how much something has increased (or decreased) from one time to the next. In these cases the part is the amount changed and the whole is the original amount.

Example 2.1.10 In spring there were 22 students in a class. In the following fall there were 34 students in the same class. What was the percent increase? Round the percent to the nearest unit.

This was an increase of 34-22=12 students. We can calculate what percent the increase of 12 is with respect to the original (spring) class size of 22.

$$100 \times \frac{12}{22} \approx 54.545454 \approx 55\%.$$

▲

We say that the class size had a **percent increase** of 55%. Note this says the **increase** was 55% of the previous **whole**.

We can think of this in another way.

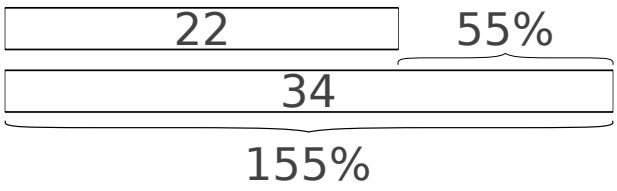
Example 2.1.11 In spring there were 22 students in a class. In the following fall there were 34 students in the same class. What is the percent increase in the fall?

We calculate the percentage the fall class size is with respect to the spring class size.

$$100 \times \frac{34}{22} = 155\%.$$

Because the fall class size (in the role of "part") is greater than the spring class size (in the role of whole), the percent ends up being greater than 100%. For percent increase we should always expect a percent greater than 100%.

Because this is 55% greater than 100%, the percent increase was 55% over the previous semester.



▲

Checkpoint 2.1.12 What is the percent increase if enrollment in a class was 55 in spring and 89 in the following fall? _____

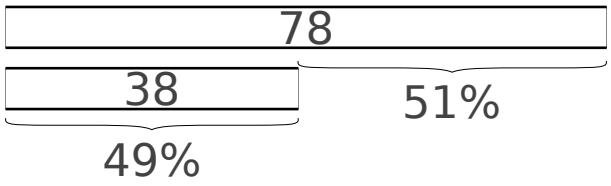
Round to the nearest percent (ones).

Example 2.1.13 What is the percent increase or decrease if enrollment in a class was 78 in fall and 38 in the following spring? Round the percent to the nearest unit.

Because 38 is less than 78 this is a decrease. Similar to the percent increase we can calculate the decrease first and then calculate the percent. $78 - 38 = 40$. Thus the percent decrease is

$$100 \cdot \frac{40}{78} \approx 0.51282051 \approx 51\%.$$

As with the percent increase we can also start by simply computing what percent the fall enrollment is with respect to the prior spring enrollment. The ratio is $100 \cdot \frac{38}{78} \approx 0.48717949 \approx 0.49$. Because the new enrollment *is* 49% of the previous enrollment the decrease is 100%-49%=51%.



▲

For percent increase problems it may seem fastest to calculate new/old-100% whereas for percent decrease problems it may seem most effective to calculate (new-old)/old. The methods perform the same calculation, so you may choose the one with which you make the correct calculation the most frequently.

Checkpoint 2.1.14 Suppose enrollment in a class was 50 in the fall and 72 in the following spring.

It was a _____ percent

1. decrease
2. increase

The following example starts with the percent increase information, and determines the original amount.

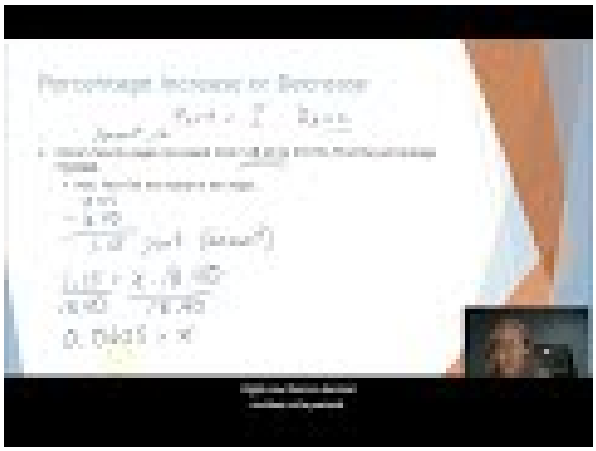
Example 2.1.15 This year's sales increased 3% over last year's sales. If total sales this year were $576,211, how much were last year's sales?

We can use the usual part/whole definition. It is important to place our information in the correct locations. We are given a percent increase, so we will want to calculate the percent. This is $100\% + 3\% = 103\% = 1.03$. The amount we are given (this year) is the part; we want to calculate the whole.

$$\begin{aligned}
\frac{\$576211}{W} &= 1.03. \\
\$576211 &= 1.03W. \text{ Clear the denominator} \\
\frac{\$576211}{1.03} &= W. \text{ Divide to isolate } W \\
\$559428.1553 &= W. \\
\$559,428.16 &= W.
\end{aligned}$$

If we want to know what amount constituted the 3% increase of the previous year, we can now calculate that as the difference. $576,211−$559,428.16=$16,782.84. We can confirm it is 3% by dividing $\frac{\$16782.84}{\$559428.16} \approx 0.029999991$ or 3% when rounded. Notice we needed to compare to the previous year's amount which was not given. ▲

This video covers percent increase topics.



Standalone
In Context

2.1.3 Limitations

We use percents because they can make the difference in scale between two quantities clear to us. However presenting a percent by itself can be deceptive.

Example 2.1.16 Which of the following do you suppose represents a greater reduction in students?

Percent reduction	Total
18%	495
1.85%	54
60%	5

Solution.

Percent reduction	Total	Number reduced
18%	495	90
1.85%	54	1
60%	5	3

The 18% of 495 represents the largest number of students. The 60% is a higher percent, but because the total is so small it represents very few students. A percent is more useful if we also know the total number.

Did you calculate 89 for 18% of 495? Compare the following to see why both are reasonable responses. 89 is what percent of 495? 90 is what percent of 495? ▲

While percent is defined a parts per one hundred, there are times when percents, sensibly, add to more than one hundred.

Example 2.1.17 Table 2.1.18 contains data from the 2020 U.S. Census. It contains the percent of the state population who checked the box for that race. Note the total is 149.4%. The reason is that a person can select more than one race. As a result a large number of people are counted more than once. Naturally the total is greater than 100 as a result.

When interpreting percents and data in general we should ask about the assumptions are before we draw conclusions. ▲

Table 2.1.18 Declared Race in Alaska

Race	Percent
Alaska Native/Native American	21.9%
Native Hawaiian/Pacific Islander	2.5%
Asian	8.4%
Black	40.8%
White	70.4%
Other	5.4%

2.1.4 Exercises

Exercise Group. Questions about the definition, terminology, and notation.

1. The decimal 0.97 is equivalent to what percent?
____ %
(Do not enter the % sign)

2. The fraction $\frac{1}{5}$ is equivalent to what percent? _____ %
(Do not enter the % sign)

3. 23 is what percent of 20? ____%

Exercise Group. Use percentages in various settings.

4. There are 15000 students attending the community college. Find the percent of students that attend classes in the evening if there are 4683 evening students.
_____ %
Round to units. Do not type the %

5. There are $16,000$ students attending a private college. There are $7,200$ evening students. What percent of all students are the evening students?
Evening students are _% of all the college's students.

6. If the first chapter of a certain book is 26 pages long and makes up 2% of the book, how many pages does the entire book have?
_____ pages

7. This week Aponi got a promotion at work that came with a 2% pay increase. If now her monthly salary is $2,040.00, how much was she making before the raise?
Enter your answer as a decimal. If needed, round to the nearest penny. Do not use commas.
$ _____

8. Joey is running a race. He has completed 4 km, which happens to be 40% of the total race. How long is the race? _____ km

9. Agustín's car insurance bill this year has decreased by 11%. He will be paying $97.57 less this year than last. How much did he pay last year?
_____

10. Out of the last 48 days, it rained 9 days. What percent of these days did it rain? _____ %
Round to 2 decimal places.

11. In 25.71 % of the last 35 days it rained. How many days did it not rain? _____ days
Round to the nearest whole.

12. An e-book regularly costs $32.99. How much does it cost if it's on sale for 63% of the regular cost?
$_____
Round to the nearest cent.

13. Laura invested $6,000 in stocks and later sold them for $7,400. The sale price is what percent of the purchase price? _____ %
Round to 2 decimal places.

14. A city has a population 280,000 in a state with population 6,500,000. What percent of the state live in this city?
_____ %
Round to 2 decimal places.

15. Identify as an increase or decrease. Then find the percent of increase or decrease. If necessary, round to the nearest percent.

Original: 90
New: 130
_____%

1. Increasing

2. Decreasing

Percents in Applications. Use percents to answer these questions in various contexts.

16. After a 30% reduction, you purchase a new suit for $329. What was the price of the suit before the reduction?

A) First write an equation you can use to answer this question. Use x as your variable and express any percents in decimal form in the equation.

The equation is _____

B) Solve your equation in part [A] to find the original price of the suit.

Answer: The original price of the suit was _____ dollars.

17. The radiator in a car is filled with a solution of 75% antifreeze and 25% water. The manufacturer of the antifreeze suggests that for summer driving, optimal cooling of the engine is obtained with only 50% antifreeze.

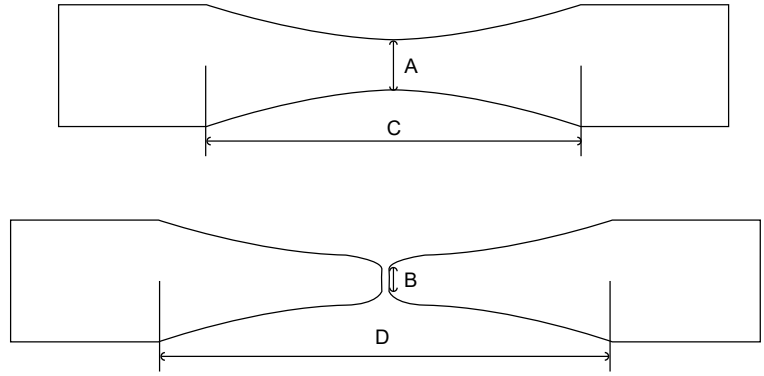
If the capacity of the radiator is 4.4 liters, how much coolant (in liters) must be drained and replaced with pure water to reduce the antifreeze concentration to 50%? Round to the nearest hundredth of a liter. ____ L

18. 10.5 liters of fuel containing 2.4% oil is available for a certain two-cycle engine. This fuel is to be used for another engine requiring a 5.3% oil mixture. How many liters of oil must be added?

Give your answer to 3 significant digits.

_____

19.



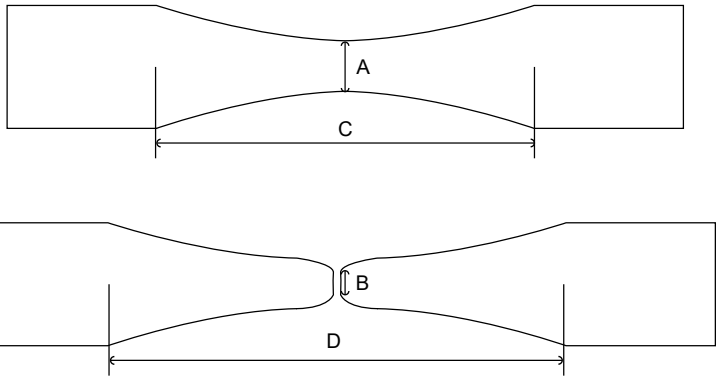
Elongation measures the amount a metal sample stretches under carefully applied force. It is used to measure how ductile (stretchable) that type of metal is. A cylinder of metal is pulled in line with the axis of the sample. The elongation is a measure of how much the sample changes after it is pulled.

$E = \dfrac{D - C}{C} \times 100$. C is the original gage length, and D is the gage length after the pull. The times 100 is to make it a percent.

If $C = 6$ and $D = 14$ what is the elongation? _____ %

Round to the nearest percent.

20.



Elongation measures the amount a metal sample stretches under carefully applied force. It is used to measure how ductile (stretchable) that type of metal is. A cylinder of metal is pulled in line with the axis of the sample. The elongation is a measure of how much the sample changes after it is pulled.

$R = \dfrac{A - B}{A} \times 100$. A is the original area, and B is the area after the pull. The times 100 is to make it a percent.

If $A = 11$ and $B = 5$ what is the reduction in area? _____ %

Round to the nearest percent.

Exercise Group. Check your understanding of percents on these quantitative literacy questions.

21. A man owns stock valued at \$2,600.00. One day the stock drops by 9%, and the next day the gain is the same percentage. How much money is the stock valued at the end of the second day? \$____

22. Enrollment in a class is 22 in the spring semester and is 13 the following fall semester. Without calculating the percent (you do not need it), what percent is the fall enrollment with respect to the spring semester?

1. less than 100%
2. 100%
3. greater than 100%

23. Enrollment in a class is 22 in the spring semester and is 13 the following fall semester. Without calculating the percent (you do not need it), what percent is the fall enrollment with respect to the spring semester?

1. less than 100%
2. 100%
3. greater than 100%

24. In which range is 34% of 112 found? This can be determined without calculating the percent.

1. 0 and 112
2. 112 and 224
3. 224 and 336

25. In which range is 34% of 112 found? This can be determined without calculating the percent.

1. 0 and 112
2. 112 and 224

3. 224 and 336

26. Which of the following is largest amount (rather than largest percent)?

1. 37% of 110
2. 99% of 87
3. 5% of 1975

27. An advertisement for Clorox contained the following statements. We will investigate the last claim.

The new Clorox® Disinfecting Bleach formula contains 7.5% sodium hypochlorite, compared to 6.0% in the previous formulation. Concentrating our formula allows us to make smaller, more easily poured bottles, reduces plastic packaging and cuts down on water waste in manufacturing. It also enables consumers to use less bleach to achieve the same results, giving you more uses per ounce.

The CDC suggests using 1/3 cup of bleach product in one gallon of water for disinfecting. If that bleach is 6.0% sodium hypochlorite, how much sodium hypochlorite is recommended? ____ cups

If this is a good amount of sodium hypochlorite, how many cups of 7.5% bleach would be needed for the same concentration? ____ cups

Is this less than the 1/3 cup of 6.0% bleach?

1. yes
2. no

How many gallons of bleach solution can one make with a 24 oz bottle of 6.0% Clorox bleach? ____

with a 24 oz bottle of 7.5% Clorox bleach? ____

Do we get more uses per ounce as claimed?

1. yes
2. no

Are we using less sodium hypochlorite?

1. yes
2. no

For a fixed number of cleanings (e.g., making the solution 45 times) will we dispose of less packaging (empty bottles)?

1. no
2. yes

2.2 Mixtures

This section addresses the following topics.

- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Calculate Percentages (skill)
- Understand and interpret percentages (critical thinking)

This section continues the topic of percents through a set of applications.

There are many situations where we desire to mix two or more substances together in precise ratio. These include mixing medicines in substances (like water) and mixing ingredients in a recipe. This section presents how to calculate the ratio of substances after mixing multiple solutions, and the reverse problem how to calculate the amount of each solution to mix for a desired ratio of substances.

2.2.1 Calculate the Result

In some circumstances we know the concentrations of substances in multiple solutions and how much of each has been combined. From this we can calculate the concentration of substances in the resulting solution.

2.2.1.1 Mix Multiple Solutions

The first mixture problem we consider is calculating the result of mixing two, known solutions.

Example 2.2.1 Suppose we have a container with a solution of sugar water that is 22% sugar and another container with a solution of sugar water that is 16% sugar. If we combine 150 g of the first solution and 250 g of the second solution, what is the percent of sugar in the resulting solution? Round the percent to two decimal places.

To calculate a percent we need the amount of the part and the amount of the whole. We are told the total amount of each solution we are using, so we can calculate the whole (total amount of resulting solution) directly. Because they have the same units we can add.

$$T = 150 \text{ g} + 250 \text{ g} = 400 \text{ g}.$$

We cannot simply add the amounts of sugar from each because we are told the percent rather than the amount. However, we know how to convert a percent and total amount to an amount of the part. We did this in Example 2.1.5.

$$\begin{aligned} S_1 &= 150 \text{ g} \cdot 0.22 \\ &= 33 \text{ g} \\ S_2 &= 250 \text{ g} \cdot 0.16 \\ &= 40 \text{ g}. \end{aligned}$$

$$\begin{aligned} P &= 33 \text{ g} + 40 \text{ g} \\ &= 73 \text{ g}. \end{aligned}$$

Finally we can calculate the percent of sugar in the resulting mixture using the defintion of percent. The percent of sugar in the resulting mixture is

$$\frac{\text{Part}}{\text{Whole}} = \frac{73 \text{ g}}{400 \text{ g}} = 0.1825 = 18.25\%.$$

▲

Example 2.2.2 Suppose we have 11.3 lbs of 4140 steel which is a type of steel containing 40% carbon and 9.2 lbs of 4150 steel which contains 50% carbon. If we melt and mix these two metals, what is the resulting percent carbon? Round the final percent to two decimal places.

To calculate a percent we need the amount of the part and the total amount. We can calculate the total directly: $T = 11.3 \text{ lbs} + 9.2 \text{ lbs} = 20.5 \text{ lbs}$. Note we know we can add these because both represented total amounts and they have the same units (g).

To calculate the part we need to know how much (rather than what percent) carbon was obtained from the two metals. We can calculate that from the given percents and amounts.

$$11.3 \text{ lbs} \cdot 0.40 = 4.52 \text{ lbs}.$$

$$\begin{aligned} 9.2 \text{ lbs} \cdot 0.50 &= 4.6 \text{ lbs.} \\ P &= 4.52 \text{ lbs} + 4.6 \text{ lbs} \\ &= 9.12 \text{ lbs.} \end{aligned}$$

Thus the percent of carbon in the resulting metal is $P/T = \frac{9.12 \text{ lbs}}{20.5 \text{ lbs}} \approx 0.44487805 \approx 44.49\%$. ▲

Checkpoint 2.2.3 Suppose 5.0 oz of a solution that is 70.% alcohol is mixed with 3.0 oz of a solution that is 90.% alcohol. The resulting solution is what percent alcohol? _____

2.2.1.2 Dilute a Solution

This section presents a slight variation of the mixture calculation problem. In this case we are adding only water resulting in diluting the solution.

Example 2.2.4 Dilute Alcohol. 91% isopropyl alcohol is not a good agtent for sanitzing because it evaporates so quickly that bacteria are not exposed long enough to die. As a result we want to dilute it before use. Suppose we begin with 16.0 oz of a 91.0% isopropyl alcohol and add 4.00 oz of water to this mixture. What will the percent alcohol of the resulting solution be? These are measurements, so we will round using significant digits.

We know we can calculate the percent alcohol by dividing the amount of alcohol (part) by the total amount. We also know that we are not increasing the amount of alcohol (adding only water), so we can use the original percent alcohol and original amount to calculate the amount of alcohol.

$$0.910(16.0 \text{ oz}) = 14.\underline{5}6 \text{ oz.}$$

We are adding 4.00 oz total to the original solution, thus the volume of the resulting solution is

$$16.0 \text{ oz} + 4.00 \text{ oz} = 20.0 \text{ oz.}$$

Now that we know the part and the whole we can calculate the resulting percent alcohol.

$$\frac{14.\underline{5}6 \text{ oz}}{20.0 \text{ oz}} = 0.72\underline{8} = 72.8\%.$$

▲

If we added more water would the percent alcohol be greater or less? If we are using all of the alcohol solution, the amount of water we add determines the percent alcohol.

Checkpoint 2.2.5 Suppose 7.4 cups of lemonade contains 18% lemon simple syrup. If 4.0 oz of sparkling water is added, what is the percent lemon in the resulting drink? _____%

2.2.2 Producing a Desired Solution

In the previous section we calculated the result of mixing two solutions. In this section the goal is to figure out how much water to add to achieve a specific concentration. That is, the previous section calculated in this section we solve.

Example 2.2.6 Calcluate Dilution. If we start with 16.0 oz of 91.0% alcohol solution, how much water do we add to obtain (at least) 25.0 oz of a 55.0% alcohol solution? These are measurements, so we will round using significant digits.

The concentration (55.0%) is the primary goal here rather than the resulting amount (25.0 oz), because we need *at least* that amount: more is okay. As a result we will set up our calculation based on the desired percent alcohol.

This is a percent problem with the total alcohol unchanged and adding only some amount w of water. This matches the previous example, except that we are calculating the amount to add and know the resulting percent. However, we can write essentially the same equation. $\frac{\text{part}}{\text{whole}} = \frac{A}{16.0+w} = 0.550$ where A is the amount of alcohol. We do not use the 25.0 oz at this time. We will address that at the end.

Because we are adding only water, all of the alcohol comes from the initial solution. Thus we can calculate the amount of alcohol in the same way as before. $(0.910)16.0 \text{ oz} = 14.56 \text{ oz}$. Thus we setup

$$\begin{aligned}
\frac{(0.910)16.0 \text{ oz}}{16.0 \text{ oz} + w \text{ oz}} &= 0.550. \\
\frac{14.\underline{5}6 \text{ oz}}{16.0 \text{ oz} + w \text{ oz}} &= 0.550 \text{ Multiply to move } w \text{ out of the denominator} \\
14.\underline{5}6 \text{ oz} &= 0.550(16.0 \text{ oz} + w \text{ oz}). \text{ Distribute} \\
14.\underline{5}6 \text{ oz} &= (0.550)16.0 \text{ oz} + (0.550)w. \text{ oz} \\
14.\underline{5}6 \text{ oz} &= 8.80 \text{ oz} + (0.550)w. \text{ oz Subtract to isolate } w \\
5.\underline{7}6 \text{ oz} &= (0.550)w \text{ oz. Divide to isolate } w \\
\frac{5.\underline{7}6 \text{ oz}}{0.550} &= w. \\
1\underline{0}.47272727 \text{ oz} &\approx w. \\
10 \text{ oz} &\approx w.
\end{aligned}$$

This means that to obtain a solution with 55.0% alcohol, we need to add 10 oz of water.

We end up with $16 \text{ oz} + 10 \text{ oz} = 27 \text{ oz}$ of new solution. Because we have at least the 25 oz we need, and it is the correct concentration, we have achieved our goal. If we had added less water to get exactly 25 oz we would have had a more concentrated solution than desired.

Also note that the percent alcohol is $\frac{0.910(16.0 \text{ oz})}{16.0 \text{ oz}+10 \text{ oz}} = 0.56$ or 56%. This is not exactly 55% because of the rounding in the calculations. If we needed it to be closer to 55%, we would need to be able to measure more precisely. ▲

Checkpoint 2.2.7 If we have 13.0 oz of a 91.0% alcohol solution left in a bottle and we want 19.0 oz of a 60.0% alcohol solution, how much water should we add? _____



Standalone
In Context

2.2.3 Exercises

1. You need a 10% alcohol solution. On hand, you have a 40 mL of a 40% alcohol mixture. How much pure water will you need to add to obtain the desired solution?

A) Write an equation using the information as it is given above that can be used to solve this problem. Use x as your variable to represent the amount of pure water you need to use. Equation:

_____

B) You will need
_____ mL of pure water
to obtain
_____ mL of the desired 10% solution.

2. 10.5 liters of fuel containing 2.4% oil is available for a certain two-cycle engine. This fuel is to be used for another engine requiring a 5.3% oil mixture. How many liters of oil must be added?

Give your answer to 3 significant digits.

_____

3. How many mL of sodium hydroxide are required to prepare 900 mL of a 15% solution? Assume the sodium hydroxide dissolves in the solution and does not contribute to the overall volume.

_____ mL

4. Quinidine gluconate is a liquid mixture, part medicine and part water, which is administered intravenously. There are 60.0 mg of quinidine gluconate in each cubic centimeter (cc) of the liquid mixture. Dr. Martinz orders 204 mg of quinidine gluconate to be administered daily to a patient with malaria.

How much of the solution would have to be administered to achieve the recommended daily dosage?

_____ cc

5. Albuterol is a medicine used for treating asthma. It comes in an inhaler that contains 15 mg of albuterol mixed with a liquid. One actuation (inhalation) from the mouthpieces delivers a 90 μg dose of albuterol. (Reminder: 1 mg = 1000 μg .)

a.) Dr. Olson orders 2 inhalations 4 times per day. How many micrograms of albuterol does the patient inhale per day?

_____ μg

b.) How many actuations are contained in one inhaler?

_____ actuations

c.) Precious is going away for 7 months and wants to take enough albuterol to last for that time. Her physician has prescribed 2 inhalations 4 times per day. How many inhalers will Precious need to take with her for the 7 period? Assume 30-day months.

Hint: she can't bring a fraction of an inhaler, and she does not want to run out of medicine while away.

_____

6. Amoxicillin is a common antibiotic prescribed for children. It is a liquid suspension composed of part amoxicillin and part water.

In one formulation there are 175 mg of amoxicillin in 9 cubic centimeters (cc's) of the liquid suspension. Dr. Scarlotti prescribes 525 mg per day for a 2-yr old child with an ear infection.

How much of the amoxicillin liquid suspension would the child's parent need to administer in order to achieve the recommended daily dosage?

_____ cm^3

7. Diphenhydramine HCL is an antihistamine available in liquid form, part medication and part water. One formulation contains 15 mg of medication in 7 mL of liquid. An allergist orders 60-mg doses for a high school student. How many milliliters should be in each dose?

_____ mL

8. You are asked to make a 1/9 dilution using 8 mL of serum. How much diluent do you need to use?

_____ mL

9. A clinical lab technician determines that a minimum of 65 mL of working reagent is needed for a procedure. To prepare a $\frac{1}{7}$ dilution of the reagent from a stock solution, one should measure 65 mL of the reagent and _____ mL of the diluent.

10. A patient's glucose result is suspected to be outside the range of the analyzer, so the techs decide to dilute the sample before running it. 40 microliters of serum is added to 120 microliters of diluent and the diluted sample is analyzed. The analyzer reads that the glucose value of the diluted sample is 31 $\frac{mg}{dL}$.

What was the ratio the sample was diluted to?

_____

What is the glucose value of the original sample?

_____ $\frac{mg}{dL}$

11. A thyroid peroxidase antibody test was performed on a 45 year old man. The dilution sequence was 15 μL serum added to 120 μL of diluent in tube 1. Then 60 μL from tube 1 was added to 240 μL of diluent in tube 2. Finally 45 μL from tube 2 was added to 225 μL of diluent in tube 3.

All dilution factors should be given as fractions.

a.) What is the dilution ratio in tube 1? Your answer should be in the form of $\frac{1}{\text{number}}$.

_____

b.) What is the ratio of the liquid from tube 1 in tube 2 after they are mixed?

_____

c.) What is the ratio of the liquid from tube 2 in tube 3 after they are mixed?

_____

d.) What is the ratio of the original serum in tube 3? That is, what is the overall (serial) dilution ratio?

_____

2.3 Ratios

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Set up and solve proportions (skill)

A ratio expresses a fixed relationship between two quantities. This section illustrates using ratios to calculate amounts subject to a ratio and presents how to recognize what a ratio does and cannot tell us. Note, percents are simply ratios scaled to parts per 100, that is, what we know about percents is applicable here.

2.3.1 Example Ratios

Example 2.3.1 Simple syrup consists of one cup of sugar and one cup of water which is heated until the sugar is disolved. There are multiple ratios that express this combination. $\frac{\text{1 cup sugar}}{\text{1 cup water}}$, $\frac{\text{7.05 oz}}{\text{8 oz}}$ ratio of sugar to water by weight, $\frac{\text{7.05 oz}}{\text{15.05 oz}}$ ratio of sugar to simple syrup.

Each of these ratios indicates that there is a fixed amount of sugar relative to the water or resulting syrup. That is, is if we made simple syrup with 2 cups of sugar we would need 2 cups of water because

$$\frac{\text{2 cups sugar}}{\text{2 cups water}} = \frac{\text{1 cup sugar}}{\text{1 cup water}}$$

by reducing fractions. ▲

Example 2.3.2 In a neighborhood there are 7 dogs and 12 cats. To express the relative number dogs and cats we can write the ratio

$$\frac{\text{7 dogs}}{\text{12 cats}}$$

Note that $\frac{7}{12} \approx 0.58 < 1$. Because it is less than one the ratio tells us that there are fewer dogs than cats in this neighborhood.

We could also write $\frac{\text{12 cats}}{\text{7 dogs}}$ to express the exact same relationship. Note that $\frac{12}{7} \approx 1.7 > 1$. Because it is greater than one the ratio tells us that there are more cats than dogs in the neighborhood (same result as above). ▲

Example 2.3.3 Rates are expressed as ratios. The following rates are all written as ratios.

$$\frac{35 \text{ miles}}{1 \text{ hour}}$$

$$\frac{8 \text{ gallons}}{1 \text{ minute}}$$

Notice that these (and most rates) are expressed with the denominator being one. This makes it easier to calculate as will be seen below. Frequently we will skip writing the one in the denominator, e.g., $\frac{35 \text{ miles}}{\text{hour}}$.

Rates are not required to have a denominator of one and sometimes a different denominator is easier for calculation. For example

$$\frac{3 \text{ nm}}{2\text{min}} = \frac{1.5 \text{ nm}}{1 \text{ min}}$$

▲

2.3.2 Using Ratios

Just like percents (which are ratios written in decimal form) if we know a ratio and one of the amounts we can calculate the other amount. The method is the same as with percents, namely multiplying the correct number or solving an equation.

Example 2.3.4 Ratio: Airspeed. According to the airplane flight manual the Diamond DA-20 cruises at the rate (speed) of

$$\frac{110 \text{ nautical miles}}{1 \text{ hour}}.$$

Cruise is a portion of flight in which the speed is typically constant. How far can the plane cruise in 2.5 hours?

We notice that the speed units (nautical miles per hour) match with the hours. If the speed and time are multiplied we will end up with a distance. Thus the units suggest we can multiply.

$$\begin{aligned}
\text{Distance} &= \frac{110 \text{ nm}}{1 \text{ hr}} \cdot 2.5 \text{ hr} \\
&= \frac{110 \text{ nm}}{1 \cancel{\text{hr}}} \cdot \frac{2.5 \cancel{\text{hr}}}{1} \\
&= 110 \cdot 2.5 \text{ nm} \\
&= 275 \text{ nm}.
\end{aligned}$$

▲

There is another approach to the same question.

Example 2.3.5 Ratio: Airspeed by Table. According to the airplane flight manual the Diamond DA-20 cruises at the rate (speed) of

$$\frac{110 \text{ nautical miles}}{1 \text{ hour}}.$$

How far can the plane cruise in 2.5 hours?

Because this ratio remains the same during cruise we can break the problem down into pieces. Based on the ratio (speed) in the first hour the plane will fly 110 nm. In the second hour it will fly another 110 nm. In the last half hour it will fly half of the distance which is $110/2 = 55$ nm. Thus in 2.5 hours it will fly $110 + 110 + 55 = 275$ nm.

Some manuals provide tables to make this method convenient. See Table 2.3.6 for a table of this sort. How far can the DA-20 cruise in 2.7 hours? We need to write 2.7 as the sum of numbers in the table. One way is $2.7 = 2 + 0.5 + 0.2$. From the table we know it will fly 220 nm in 2 hours, 55 nm in an additional 0.5 hours, and 22 nm in the final 0.2 hours. Thus it will fly $220 + 55 + 22 = 297$ nm.

While this method makes sense, it is less efficient than simply multiplying as in the previous example.

▲

Table 2.3.6 Airspeed and Distance

Time	Distance
0.1 hour	11 nm
0.2 hour	22 nm
0.3 hour	33 nm
0.4 hour	44 nm
0.5 hour	55 nm
1 hour	110 nm
2 hour	220 nm
3 hour	330 nm

Example 2.3.7 Water is flowing out of a hose at a rate of 11 gallons per minute. How many gallons have come out after 2.7 minutes?

Solution. We can set this up like the calculation in Example 2.3.4. In that example multiplied the ratio (nm/hr) by the number of hours which gave us nm. Here we multiply the ratio (gal/min) by the number of minutes which will give us gallons.

$$\begin{aligned}\text{Volume} &= \frac{11 \text{ gal}}{1 \text{ min}} \cdot 2.7 \text{ min} \\ \text{Volume} &= \frac{11 \text{ gal}}{1 \cancel{\text{ min}}} \cdot \frac{2.7 \cancel{\text{ min}}}{1} \\ &= 11 \cdot 2.7 \text{ gal} \\ &= 29.7 \text{ gal.}\end{aligned}$$

▲

Example 2.3.8 Based on data from the FDA the average amount of mercury found in fresh or frozen salmon is 0.022 ppm (parts per million). This means there are 0.022 mg of mercury in one liter of salmon. If a meal portion of salmon is 0.0020 liters how much mercury is consumed?

Solution. We can use the ratio $\frac{0.022 \text{ mg}}{1 \ell}$. We apply this ratio to the given volume of 0.0020 liters.

$$\frac{0.022 \text{ mg}}{1 \ell} \cdot 0.0020 \ell = 0.000044 \text{ mg}$$

▲

Checkpoint 2.3.9 Calculate how far a plane flying at 150 nm/hour would travel in 2.1 hours.

The method for ratios that have a number other than one in the denominator is the same.

Example 2.3.10 A saline solution intended for nasal rinsing has a ratio of 2.5 g of salt (sodium chloride) per 240 mL of pure water. How much salt is needed to make a half liter of this saline solution? Your scale is precise to a gram. Round appropriately.

We can apply the given ratio (2.5 g/240 mL) to the given amount (0.5 L). First it will be convenient to convert a half liter to milliliters. This is also a ratio problem. See Table 1.1.8 for the conversion ratio.

$$\frac{1000 \text{ mL}}{1 \text{ L}} \cdot 0.5 \text{ L} = 500 \text{ mL}.$$

The units suggest that we can multiply the saline solution ratio by the desired volume.

$$\frac{2.5 \text{ g}}{240 \text{mL}} \cdot 500 \text{ mL} \approx 5.208333335 \approx 5 \text{ g}.$$

We round to units, because we cannot measure more precisely. ▲

Example 2.3.11 At 90 nm/hr a plane travels 3 nm/2 min. How far will it travel in 6 minutes?

Solution. We can multiply the given ratio (nm/hr) by the given amount (min) to calculate distance (nm).

$$\frac{3 \text{ nm}}{2 \text{ min}} \cdot 6 \text{ min} = 9 \text{ nm}.$$

▲

Checkpoint 2.3.12 One formulation of amoxicillin, a drug used to treat infections in infants, contains 125 mg of amoxicillin per 5.00 mL of liquid. How much amoxicillin is in 11.0 mL of liquid? _____

2.3.3 Understanding Ratios

Like percents ratios tell us a relationship between two quantities but do not tell us how much. For example if a cookie recipe calls for 2 cups of milk for every 3 eggs, we do not know how many eggs are needed for a dozen cookies. We would also need to know either how many cups of milk per dozen or how many eggs per dozen.

Ratios may be based on rounded numbers. For example the unit convesion $\frac{8 \text{ kilometers}}{5 \text{ miles}}$ is convenient for quick calculations. However, $8/5 = 1.6 \approx 1.609344$ which is a more accurate conversion rate. If we are trying to convert "Tempo 30" (a speed limit of 30 kph) to mph this ratio is fine. If we are sending a satellite to another planet, we will need a more accurate conversion.

Rounding may occur to make the ratio easier to comprehend. For example according to an article by the U.S. Census Bureau 1 in 6 people in the U.S. was aged 65 and over. Because this ratio uses small numbers it is easy to understand. It is much easier to read and use than a more precise estimate of $\frac{57822315}{333287562}$.

2.3.4 Exercises

Writing Ratios. Write the ratios requested.

1. Enter the ratio as a fraction in lowest terms
4 ft to 84 in.
_____

2. Enter the ratio as a fraction in lowest terms.
28 minutes to 20 minutes
_____

3. Write the ratio as a ratio of whole numbers in lowest terms.
\$1.00 to \$0.60
_____

4. Enter the ratio as a fraction in lowest terms (no decimals).
If the answer can simplify to a whole number, write it as such.
0.5 miles to 0.01 miles
_____

Ratios in Contexts. Read the description, and write the requested ratio.

5. Consider the rectangle with width 27 in and length 39 in, write a ratio of the length to the width.
_____

6. If you spend 6.5 hours a week studying for English and 1.5 hours studying for math what is the ratio of time spent studying in math to studying for English?
_____

7. An employee pays \$125 towards health insurance, while the employer pays \$350. What is the ratio of the employers contribution to the employees contribution?
_____

8. At a recent Pac-12 sporting event, there were 26,000 Ducks fans and 20,800 Utes fans. Write each ratio as as a reduced fraction.

A) The ratio of Ducks fans to Utes fans. _____

B) The ratio of Ducks fans to total fans. _____

9. In a recent survey, 50 percent of people claimed to like math. Write this ratio as a reduced fraction.

_____

10. Aponi spent 14 hours playing video games in a week. The table shows the number of hours Aponi played each video game.

video games	hours:
Battlefield	5
Minecraft	3
Call of Duty	6

What is the ratio of Minecraft to total hours Aponi plays video games in a week?

__ : __

Using Ratios in Contexts. Answer the question using ratios.

11.



17 ounces = 481 grams.Use that conversion factor to determine the weight in grams of a 40 ounce box of Cherrios.

Round your answer to the nearest whole gram. 40 ounces = _____ grams

12.



12 fluid ounces (fl oz),355 milliliters (ml).In Botswana they sell 200 ml cans. How many fluid ounces is that?

*Round your answer to 2 decimal places.*200 ml =___fl oz

13. A few winters ago, it was very cold in northern Wisconsin, and Mackenzie thought about leaving the kitchen faucet running overnight (so the water pipes wouldn't freeze). Mackenzie's roommate Reza was a little concerned that they would be wasting a lot of water, so they performed an experiment.

Reza turned the kitchen faucet on so it was dripping water at a constant rate. Then he held up a 1/4 teaspoon under the faucet, and it filled in 9 seconds. So, the water was "flowing" at a rate of 0.25 teaspoons per 9 seconds.

Question: They were going to sleep and planned to get up 10 hours later to turn off the faucet. How many *gallons of water* would have gone down the drain in that time?

[Answer this question by converting 10 hours into gallons. Give your answer as a decimal number, rounded correctly to the nearest thousandth of a gallon.]

- 2 tablespoons = 1 fluid ounce
- 1 hour = 60 minutes
- 3 teaspoons = 1 tablespoon
- 1 gallon = 128 fluid ounces
- 60 seconds = 1 minute
- 9 seconds = 0.25 teaspoons

In all, approximately _____ gallons of water will flow down the drain in 10 hours.

14. A 12 oz bottle of dish soap sells for $2.41. A 35 oz bottle of dish soap sells for $9.66.
(round all answers to four decimal places)
The unit price of the 12 oz bottle is $____ per oz
The unit price of the 35 oz bottle is $____ per oz
Which of the two is a better deal?

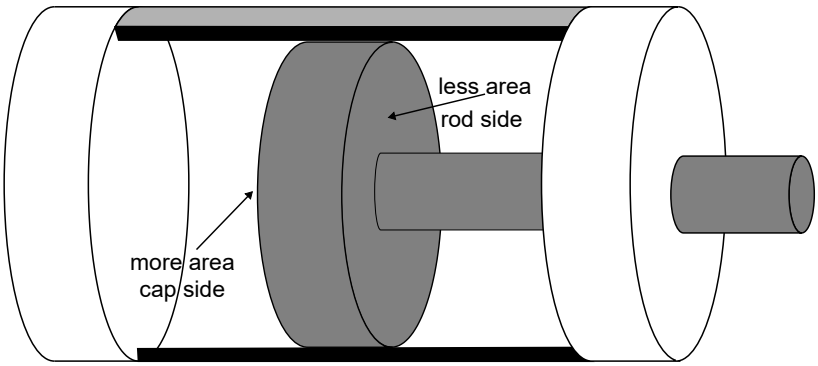
1. The 12 oz bottle for \$2.41
2. The 35 oz bottle for \$9.66

15. You can purchase a 13 fl oz bottle of window cleaner for \$2.3 or a 16 fl oz bottle for \$2.86. Which bottle of window cleaner is the better deal? What is the unit price of this bottle?

1. 16 fl oz bottle for 17.88 cents per fl oz
2. 16 fl oz bottle for 5.59 cents per fl oz
3. 13 fl oz bottle for 17.69 cents per fl oz
4. 13 fl oz bottle for 5.65 cents per fl oz
5. None of these

16. A hyraulic actuator converts hydraulic pressure into linear motion. It consists of a hollow cylinder in which a piston moves up and down. The piston consists of a puck attached to a cylindrical rod.

Because the rod end of a piston has the rod attached, the surface area on that side is less than the area on the cap side by the area of the rod. As a result more pressure is needed on the rod end to create the necessary force for motion. The amount of force needed is proportional to the ratio of cap to rod area.



If the ratio of cap to rod area is 16/9 and the pressure on the cap side is 10000, what pressure will be needed on the rod side? _____

Interpretting Ratios. Determine what each ratio tells us about the circumstances.

17. In one school classroom the ratio of families who have a dog to those who have a cat is $\frac{13}{8}$. In this class

1. fewer
2. same
3. more

18. In one school classroom the ratio of families who have a bird to those who have a bird or hamster is $\frac{14}{33}$. In this class

1. fewer
2. same
3. more

19. In one version of Snell's Law contains the ratio $\frac{V_1}{V_2}$ where the V_i are the speeds of sound in two materials (like brass and glass). The goal is to determine if the units for velocity make a difference.

Suppose $V_1 = 4300$ m/s and $V_2 = 5300$ m/s. Calculate $\frac{V_1}{V_2}$ = _____

The resulting units (of the ratio) are

1. none
2. m
3. m^2
4. s
5. s^2

1. none
2. m
3. m^2
4. s
5. s^2

Convert the units to feet per second.
V_1 =_____ ft/s
V_2 =_____ ft/s
Using the new units calculate $\frac{V_1}{V_2}$ =_____
Did changing the units change the result?

1. no
2. yes

Explain:

2.4 Proportions

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Set up and solve proportions (skill)

In some circumstances a ratio is fixed. By fixed ratio we mean that no matter which data points we select, we obtain a ratio with equal value. For example when we scale a recipe the ratio of flour to water must remain the same if we make one recipe or double it. These circumstances are called **proportions**. An example of a non-fixed ratio is a bobble head toy. The head has a different scale (ratio of toy head size to real head size) than the body does.

This section presents a variety of problems in which these arise and, indirectly, reviews the algebra needed to solve them.

Fixed ratios make sense in examples like conversion of units. For example 1 gallon is always 4 quarts. In contrast rates often change: your average speed may be 25 mph, but you must have driven slower and faster during that drive. For the ratios that do not change we can write equations and solve for properties.

2.4.1 Proportion Examples

Because a proportion means a ratio is fixed, we can write the first ratio equals the second ratio. This gives us an equation to solve. There are multiple ways to solve these, each of which is demonstrated below.



Standalone
In Context

The first example shows a straight forward proportion with the simplest solving method. This method uses the same algebra steps as solving a percent problem.

Example 2.4.1 Cheesecake Groceries: Double. A particular cheesecake recipe calls for 150 g of eggs and 1500 g of cream cheese. How many grams of eggs and how many grams of cream cheese do we need to double the recipe?

This means everything will be in ratio of 2 (needed)/1 (in the recipe). We must perform the calculation for eggs and cream cheese separately. We want the ratio of amount of eggs in the doubled recipe to the amount of eggs in the original recipe to be 2/1. Thus

$$\frac{2}{1} = \frac{E \text{ g}}{150 \text{ g}}.$$

Because the quantity to solve is in the numerator we can simply multiply to isolate that quantity (variable).

$$\begin{aligned}
\frac{2}{1} &= \frac{E \text{ g}}{150 \text{ g}}. \\
2 &= \frac{E \text{ g}}{150 \text{ g}}. \\
2 \cdot (150 \text{ g}) &= \frac{E \text{ g}}{150 \text{ g}} \cdot (150 \text{ g}). \text{ Multiply to isolate } E. \\
2 \cdot (150 \text{ g}) &= \frac{E \text{ g}}{\cancel{150 \text{ g}}} \cdot \cancel{(150 \text{ g})}. \\
2 \cdot 150 \text{ g} &= E \\
300 \text{ g} &= E \text{ of eggs.}
\end{aligned}$$

Because scales are accurate to a gram, we do not need to round.

We can perform the same calculation for the cream cheese.

$$\begin{aligned}
\frac{2}{1} &= \frac{C \text{ g}}{1500 \text{ g}}. \\
\frac{2}{1} \cdot (1500 \text{ g}) &= \frac{C \text{ g}}{1500 \text{ g}} \cdot (1500 \text{ g}). \\
2 \cdot 1500 \text{ g} &= C. \\
3000 \text{ g} &= C.
\end{aligned}$$

In commercial recipes (and quality home cooking) weights are used because items like eggs are not uniform in mass. If we always use 3 eggs (average total weight 150 g), it might be more (e.g, 152 g) or less (147 g) than we need which will result in the food not turning out right. ▲

This example may seem overly simple because doubling is easy, but the arithmetic is the same for any scaling.

Example 2.4.2 Guido needs 6 dozen cookies. A recipe makes 4 dozen. If that recipe calls for 300 g of flour, how much flour does he need for the 6 dozen cookies?

First, we determine the ratio for scaling. We want 6 dozen and the recipe makes 4 dozen so our ratio is $6/4 = 3/2$. Thus to determine the amount of flour needed we set up the proportion

$$\begin{aligned} \frac{3}{2} &= \frac{F \text{ g}}{300 \text{ g}} \\ \frac{3}{2} \cdot 300 \text{ g} &= \frac{F \text{ g}}{300 \text{ g}} 300 \text{ g} \\ 450 \text{ g} &= F \end{aligned}$$

▲

The following example shows methods for handling proportions when the desired quantity (variable) ends up in the denominator.

Example 2.4.3 Scale Model. A model of a space shuttle is labeled as 1:144. This means one inch on the model represents 144 inches on the actual shuttle. We can express this as a ratio between the size of objects on the model and the size of the objects on the actual shuttle. If we want to know how large a part of the shuttle is and we know on the model it is 1.72 inches we can set up the proportion

$$\frac{1 \text{ model inches}}{144 \text{ inch}} = \frac{1.72 \text{ model inches}}{s \text{ inch}}.$$

Because a ratio expresses the relationship between two quantities, it is not important which is numerator or denominator. Thus it is equally valid to write

$$\begin{aligned} \frac{144 \text{ in}}{1 \text{ model in}} &= \frac{s \text{ in}}{1.72 \text{ model in}}. \\ \frac{144 \text{ in}}{1 \text{ model in}} \cdot 1.72 \text{ model in} &= \frac{s \text{ in}}{1.72 \text{ model in}} \cdot 1.72 \text{ model in}. \\ 247.68 \text{ in} &= s. \end{aligned}$$

▲

The following example shows an alternate way to solve for the desired quantity when it is in the denominator.

Example 2.4.4 Proportion: Solving. According to the airplane flight manual a Diamond DA-20 cruises at the rate of

$$\frac{110 \text{ nautical miles}}{1 \text{ hour}}.$$

How long will it take to travel 236 nm? Round to tenths of an hour (a standard in aviation).

The proportion is

$$\begin{aligned} \frac{110 \text{ nm}}{\text{hr}} &= \frac{236 \text{ nm}}{t \text{ hr}}. \\ \frac{110 \text{ nm}}{\text{hr}}(t \text{ hr}) &= \frac{236 \text{ nm}}{t \text{ hr}}(t \text{ hr}). \end{aligned}$$

Multiply both sides to eliminate the division.

$$\frac{110 \text{ nm}}{\cancel{\text{hr}}}(t \cancel{\text{ hr}}) = \frac{236 \text{ nm}}{\cancel{t \text{ hr}}}\cancel{(t \text{ hr})}.$$

$$\begin{aligned}(110\text{ nm})t &= (236\text{ nm})\\ \frac{(110\text{ nm})t}{(110\text{ nm})} &= \frac{(236\text{ nm})}{(110\text{ nm})}\\ \frac{\cancel{(110\text{ nm})}t}{\cancel{(110\text{ nm})}} &= \frac{(236\ \cancel{\text{nm}})}{(110\ \cancel{\text{nm}})}\\ t &\approx 2.14545\\ t &\approx 2.1\text{ hr.}\end{aligned}$$

▲

Checkpoint 2.4.5 If the Diamond DA-20 climbs at the rate of $\frac{450\ \text{ feet}}{1.0\ \text{ minute}}$ how long will it take it to climb $4,500$ ft? _____

2.4.2 Multiple Proportions

When we experience math in the wild, problems do not come labeled with solving methods. We must recognize the math and apply our knowledge appropriately. The next example illustrates identifying ratios (proportions) more than once when answering a question.

Example 2.4.6 Suppose a Diamond DA-20 climbs at the rate of $\frac{550\text{ feet}}{1.0\text{ minute}}$ and is traveling $\frac{72.8\text{ nm}}{\text{hour}}$ across the ground during this climb. How far forward in nautical miles does the plane fly during a climb of 3500 feet? Round the final distance to one decimal place.

Looking at the information we are given, only two pieces match initially: we have a climb rate and a distance to climb. Thus we will use these first. As before we can calculate how long it will take to climb.

$$3500\text{ ft}\cdot\frac{1\text{ min}}{550\text{ ft}}\approx 6.36363636\text{ min.}$$

Now we know how long the plane flies during this climb. This time matches with the other rate: speed across the ground. However, the units do not match. We must first convert the speed to feet per minute or the time to hours. We use conversions from Table 1.1.2.

$$\frac{72.8\text{ nm}}{\text{hr}}\cdot\frac{6076\text{ ft}}{\text{nm}}\cdot\frac{1\text{ hr}}{60\text{ min}}\approx\frac{7372.213333\text{ ft}}{\text{min}}.$$

Now the units suggest that we can multiply the speed by the time we obtained, and we will end up with the desired distance.

$$\begin{aligned}\frac{7372.213333\text{ ft}}{\text{min}} &= \frac{s\text{ ft}}{6.36363636\text{ min}}\\ \frac{7372.213333\text{ ft}}{\text{min}}\cdot(6.36363636\text{ min}) &= \frac{s\text{ ft}}{6.36363636\text{ min}}\cdot(6.36363636\text{ min})\\ 46914.08482 &\approx s\text{ ft.}\end{aligned}$$

Now we need to convert this result back to nautical miles. This is a unit conversion problem again.

$$(46914.08482\text{ ft})\cdot\frac{1\text{ nm}}{6076\text{ ft}}\approx 7.72121212\text{ nm}\approx 7.7\text{ nm.}$$

▲

Example 2.4.7 A recipe for hush puppies calls for 150 g of flour for 340 g of buttermilk. If we have 465 g of flour and 918 g of buttermilk, how much of the flour and buttermilk can we use? Which one constrains us (limits size of our batch)? Quality kitchen scales are accurate to a single gram which determines our rounding.

The ingredients must remain in the ratio $\frac{340 \text{ g buttermilk}}{150 \text{ g flour}} = \frac{34 \text{ g buttermilk}}{15 \text{ g flour}}$. We can select either ingredient and see how much the ratio tells us we need of the other ingredient.

Because we do not yet know which ingredient is limiting we will suppose we use all 465 g of flour. Then we can set up the proportion

$$\frac{34 \text{ g buttermilk}}{15 \text{ g flour}} = \frac{B \text{ g}}{465 \text{ g flour}}$$
$$\frac{34 \text{ g buttermilk}}{15 \cancel{\text{ g flour}}} \cdot (465 \cancel{\text{ g flour}}) = \frac{B \text{ g}}{\cancel{465 \text{ g flour}}} \cdot \cancel{(465 \text{ g flour})}$$
$$1054 \text{ g buttermilk} = B.$$

Notice that this is more buttermilk than we have. That means the buttermilk is the limiting ingredient. We will be able to use all of the buttermilk, but only some of the flour. To determine how much we setup the proportion but this time solve for flour.

We will use all 918 g of buttermilk. Then we can set up the proportion

$$\frac{15 \text{ g flour}}{34 \text{ g buttermilk}} = \frac{F \text{ g}}{918 \text{ g buttermilk}}.$$
$$\frac{15 \text{ g flour}}{34 \cancel{\text{ g buttermilk}}} \cdot (918 \cancel{\text{ g buttermilk}}) = \frac{F \text{ g}}{\cancel{918 \text{ g buttermilk}}} \cdot \cancel{(918 \text{ g buttermilk}).}$$
$$405 \text{ g} = F.$$

Thus we can use all 918 g of buttermilk and 405 of the 465 g of flour. We have rounded everything to one gram because that is as accurate as we can measure with our scale. If a single recipe uses 340 g of buttermilk, then we will be making

$$\frac{918 \text{ g}}{340 \text{ g}} = 2.7$$

times as much. ▲

Example 2.4.8 There is an alternate way to determine which ingredient limits our scaling up. We know the recipe calls for a buttermilk to flour ratio of $\frac{34}{15} \approx 2.266666667$. We can calculate the ratio of the ingredients we have on hand. $\frac{918}{465} \approx 1.974193548$. This is less than the required ratio, which means the numerator is too small. Thus we have too little buttermilk. From here we can continue to calculate the usable amount of flour as in the previous solution. ▲

Checkpoint 2.4.9 A cheesecake recipe calls for 150 g of eggs and 750 g of cream cheese. If you currently have 200 g of eggs and 1058 g of cream cheese, determine which ingredient is the limiting one (will use all of it) and the amount of each you will use.

Limiting ingredient:

1. cream cheese
2. eggs

Eggs: _____ g Cream cheese: _____ g

2.4.3 Similar polygons

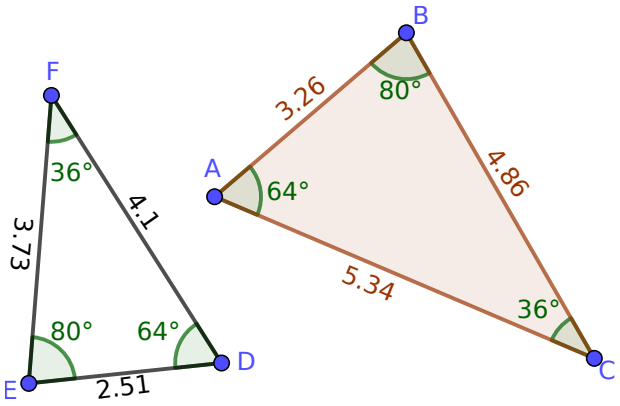
This section presents a geometric fact which is expressed as proportions. This geometry can be used to solve for distances or lengths in some circumstances. First we define and illustrate the geometric fact.

2.4.3.1 Explaining Similar Triangles

Definition 2.4.10 Similar Triangles. Two triangles are **similar** if and only if corresponding angles have the same measure. ◆

Corresponding means we have paired angles on one triangle with those on another. So this definition says that each angle on one triangle can be paired with one on the second triangle that has the same measure. This allows us to define corresponding sides as well. Two sides from the different triangles are corresponding if they are across from angles of the same measurement. When triangles are similiar their corresponding side lengths are proportional. This is illustrated in the following example.

Example 2.4.11



The triangles $\triangle ABC$ and $\triangle DEF$ are similar. Notice that the angles at A and D have the same measure (64°). The same is true of the angles at B and E (both 80°) and the angles at C and F (both 36°). These are the corresponding angles: A and D, B and E, and C and F.

BC is the side opposite the angle at A and EF is the side opposite the angle at D. Because A and D have the same measure, the sides opposite them are corresponding sides.

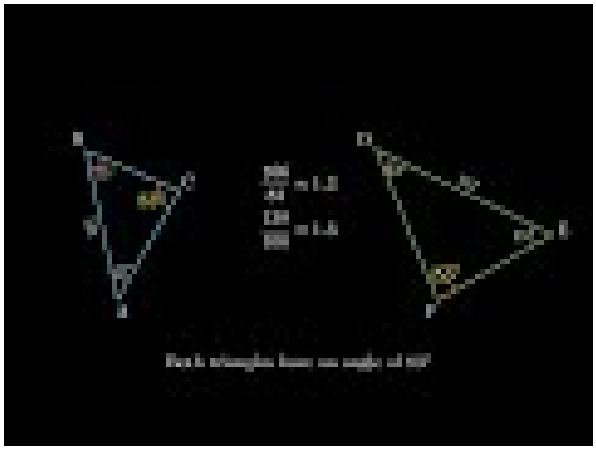
Similarly CA is the side opposite the angle at B and FD is the side opposite the angle at E. Because B and E have the same measure, the sides opposite them are corresponding sides.

What is the third pair of corresponding sides?

As a result of being similar the following ratios of sides are the same

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

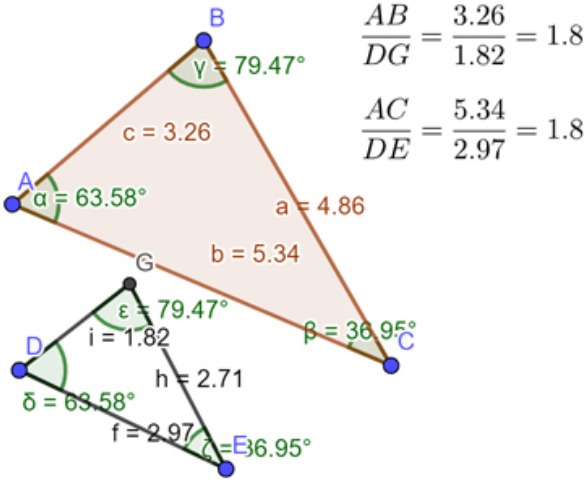
You can confirm this by dividing the lengths $\left(\frac{3.26}{2.51} = \frac{4.86}{3.73} = \frac{5.34}{4.1}\right)$. ▲



Standalone
In Context

From a single example we might think this was just a special case. To convince yourself use the following

interactive example.



$$\frac{AB}{DG} = \frac{3.26}{1.82} = 1.8$$

$$\frac{AC}{DE} = \frac{5.34}{2.97} = 1.8$$

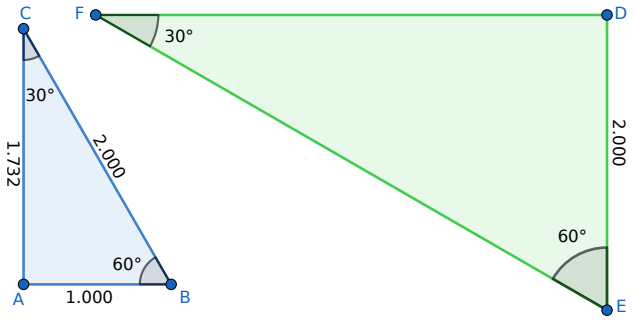
Standalone
In Context
Embed

Figure 2.4.12 Similar Triangles

2.4.3.2 Calculating Using Similar Triangles

We can use the proportionaliy of similar triangle sides lengths to calculate the lengths using the same technique as Example 2.4.4.

Example 2.4.13



Suppose triangle ABC has angles $90°, 60°, 30°$ with the lengths of the sides opposite them $2.000, 1.732, 1.000$ respectively. If triangle DEF also has angles $90°, 60°, 30°$ it is similar. Suppose the length of the side DE opposite the $30°$ angle at point F is 2.000.

First, we identify the corresponding sides. $\overline{AB}$ and $\overline{DE}$ are opposite $30°$ angles so they are corresponding. $\overline{CA}$ and $\overline{FD}$ are opposite $60°$ angles so they are corresponding. Finally, $\overline{BC}$ and $\overline{EF}$ are opposite $90°$ angles so they are corresponding. This means

$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{CA}}{\overline{FD}} = \frac{\overline{BC}}{\overline{EF}}.$$

Because we know the ratio $\frac{\overline{AB}}{\overline{DE}}$, we can use the proportion to solve for the other two side lengths on triangle DEF. We invert the ratios for easier solving.

$$\begin{aligned}
\frac{\overline{DE}}{\overline{AB}} &= \frac{\overline{DF}}{\overline{CA}} \\
\frac{2.000}{1.000} &= \frac{\overline{DF}}{1.732}. \\
\frac{2.000}{1.000} \cdot 1.732 &= \frac{\overline{DF}}{1.732} \cdot 1.732. && \text{Multiply to isolate } \overline{DF} \\
3.464 &= \overline{DF}.
\end{aligned}$$

We can calculate the length of the third side in the same way.

$$\begin{aligned}
\frac{\overline{DE}}{\overline{AB}} &= \frac{\overline{EF}}{\overline{BC}}. \\
\frac{2.000}{1.000} &= \frac{\overline{EF}}{2.000}. \\
\frac{2.000}{1.000} \cdot 2.000 &= \frac{\overline{EF}}{2.000} \cdot 2.000. \\
4.000 &= \overline{EF}.
\end{aligned}$$

▲

Checkpoint 2.4.14 Suppose triangle A has angles 60°, 90°, and 30° and the lengths of the sides opposite are 10, 11.55, and 5.77 respectively. If triangle B has the same angle measures and the side opposite the angle of measure 60° is length 20, what are the other two side lengths?

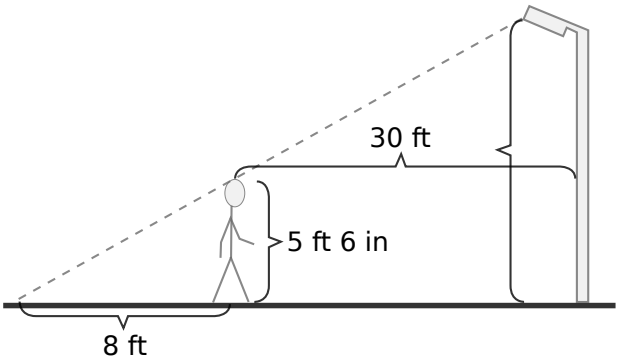
Length opposite angle of measure 90°: _____

Length opposite angle of measure 30°: _____

2.4.3.3 Similarity in Applications

Similar triangles can be found in a variety of circumstances. This example recognizes similar triangles in a context where we want to calculate rather than measure a length. This method of indirect measurement has been used many time in history.

Example 2.4.15



A person is standing 30 ft from a light pole. The shadow cast by the light is 8 ft long. If the person is 5 ft 6 in tall, how high is the point on the light that is casting the shadow?

We can start by looking at the image. In it we spot multiple triangles. Two, right triangles will be useful for calculating the height. The smaller one has legs of length 8 ft and 5 ft 6 in. The third side (hypotenuse) is the dashed gray line, but we will not need it. The other right triangle we can use has a leg that is the entire bottom (length 8 ft plus 30 ft). Its other leg is the height of the light.

Let's compare the angles of the triangles. If we suppose the light post is straight up and the person is standing straight up, then the angles at the persons feet and the base of the light are right angles and thus the same. The triangles share the angle on the left (between dashed line and ground), so that matches. The third angle must match because the first two do; we will explain why in Theorem 7.1.1.

Because all the angles in these triangles are the same, the triangles are similar. We can use the proportionality of corresponding side lengths to calculate the height of the light. Before we do, we will convert the height of the person to decimal. Using a conversion from Table 1.1.2, 5 ft 6 in is 5.5 ft. The proportion of corresponding sides is

$$\begin{aligned}
\frac{5.5 \text{ ft}}{8 \text{ ft}} &= \frac{h}{38 \text{ ft}}. \\
\frac{5.5 \text{ ft}}{8 \text{ ft}} \cdot (38 \text{ ft}) &= \frac{h}{38 \text{ ft}} \cdot (38 \text{ ft}).
\end{aligned}$$

$$26.125 \text{ ft} = h.$$

If these measurements were taken with a tape measure we can reasonably suppose they are accurate to the nearest inch. We can convert the 0.125 ft into inches and then round.

$$0.125 \text{ ft} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 1.5 \text{ in.}$$

Thus the height to that point on the light is approximately 26 ft and 2 in.

We use this type of measurement because it is simpler: we can measure across the ground much more easily than we can climb the pole and measure its height. It was not important that the person chose to be 30 ft from the pole. If they chose to be 20 ft, then the shadow would also be shorter (proportional). If we are too close it will be hard to accurately measure the short shadow (ever try to measure a shadow?). ▲

2.4.3.4 Similarity Beyond Triangles

Shapes other than triangles can be similar. For example there are similar rectangles and similar pentagons. To be similar they must have the same number of sides, corresponding angles must be the same, and corresponding sides must be in the same ratio. Just having the same angles is insufficient: any two rectangles have all the same angles (right angles) but not every pair is similar.

Similar Polygons. One way to define similar polygons that avoids the trap illustrated by rectangles is to break the shape down into triangles (e.g., a rectangle can be split into two triangles). Then we require that every such triangle has the same angles. This is complicated because how we break a shape down into triangles is not unique. Take two, non-similar rectangles and identify triangles that don't match.

One place where similar shapes (beyond triangles) is used is scale drawing and scale models. If you ever built a model of a car or a plane or some such there was most likely a scale given. For example they may be 1/32 scale. This means that one inch on the model is 32 inches on the actual object.

2.4.4 Exercises

Contextless Proportions. Solve each of these proportions.

1. Find the unknown number in the proportion.

$$\frac{x}{10} = \frac{14}{35}$$

_____

2. Solve for the variable in $\frac{x}{7.4} = \frac{2.56}{6.4}$

$x =$_____

3. Find the unknown number in the proportion.

$$\frac{2}{18} = \frac{9}{x}$$

_____

Proportions in Context. Identify a proportion in each application. Set it up, and solve for the requested value(s).

4. Cellular phone service that charges per-minute will charge \$10 for 50 minutes. How much would 1196 minutes cost?

Round your answer to the nearest cent.

\$_____

5. Gwen's Gravel Company supplied a homeowner with 18 cubic yards of gravel for his driveway at a cost of \$1,503.00. Assuming they charge the same rate per cubic yard regardless of the amount of gravel supplied, what would they charge for 32 cubic yards of gravel?

Translate to a proportion: _____
$_____ for 32 cubic yards of gravel

6. Guido is flying west at 2500 MSL using the local altimeter setting of 29.00 inHg. In the lower atmosphere the pressure drops approximately 1 inHg per 1000 feet of elevation gained.
How much does the pressure drop from the ground (0 MSL) to 2500 ft above? _____ inHg
If the pressure is 29.00 inHg at 0 MSL, what is the pressure at 2500 ft MSL? _____ inHg

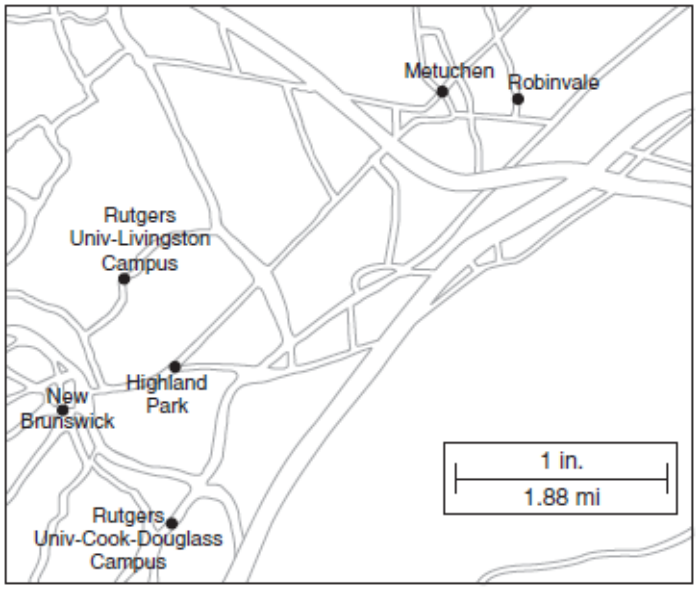
7. A recipe for lemon tea cookies calls for $1\frac{1}{2}$ cups of flour for every $\frac{3}{2}$ cup of sugar. How many cups of sugar are needed if $1\frac{2}{3}$ cups of flour are used?
For $1\frac{2}{3}$ cups of flour you need _____ cups of sugar.

8. A label reads: "2.5 mL of solution for injection contains 1,000 mg of streptomycin sulfate." How many millilitres are needed to give 800 mg of streptomycin?
_____

9. A floor plan has a 48 : 1 scale. On the drawing, one of the rooms measures $1\frac{1}{2}$ inches by $1\frac{3}{8}$inches. Round answers to the nearest hundredth.
The actual dimensions would be: _____ feet by: _____ feet.
The area of the room would be _____ ft^2.

10. While planning a hiking trip, you examine a map of the trail you are going on hike. The scale on the map shows that 2 inches represents 3 miles.
If the trail measures 8 inches on the map, how long is the trail?
_____ miles

11. Below is a map of Central New Jersey. Using an inch ruler on the map, you found the distance between Rutgers University Cook-Douglas Campus and Metuchen is $2\frac{1}{4}$ inches. Find the actual distance between Rutgers University Cook-Douglas Campus and Metuchen.



The actual distance between Rutgers University Cook-Douglas Campus and Metuchen is _____ miles.

12. A carpet store charges $385.00 to install 70 square yards of carpet. Assuming they charge the same rate per square yard regardless of the amount of carpet installed, how much would they charge to install 200 square yards of carpet? Use the variable x in setting up the proportion.
What is the unit price for installation per square yard of carpet?

Translate to a proportion: _____
They would charge \$___ to install 200 square yards of carpet.

\$___ per square yard

13. If a 30-acre alfalfa field produces 90 tons of hay, how many acres would be needed for a field to produce 153 tons of hay?

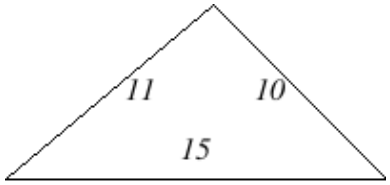
Translate to a proportion: _____
A field would need to be _____ acres to produce 153 tons of hay.

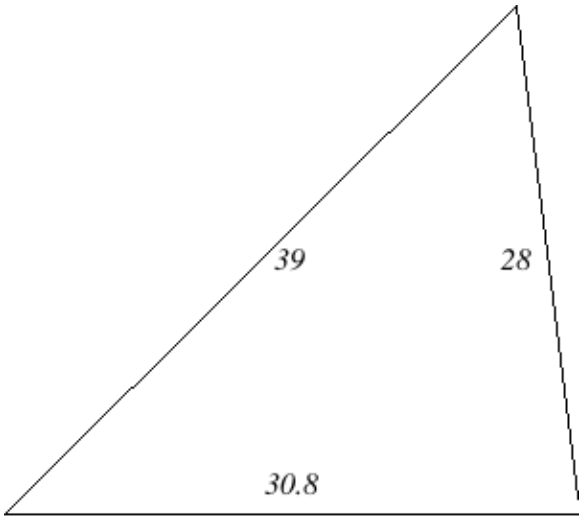
Similar Triangles. Use the property of side ratios for similar triangles to calculate the values requested.

14. The side lengths of ΔABC are: $AB = 3$ $BC = 7$ $AC = 6$
The side lengths of ΔRST are: $RS = 3$ $ST = 7$ $RT = 12$
Simplify the given corresponding side ratios:
$\frac{RS}{AB} =$ ____ $\frac{ST}{BC} =$ ____ $\frac{RT}{AC} =$ ____
Are ΔABC and ΔRST similar??

1. Yes
2. No

15.



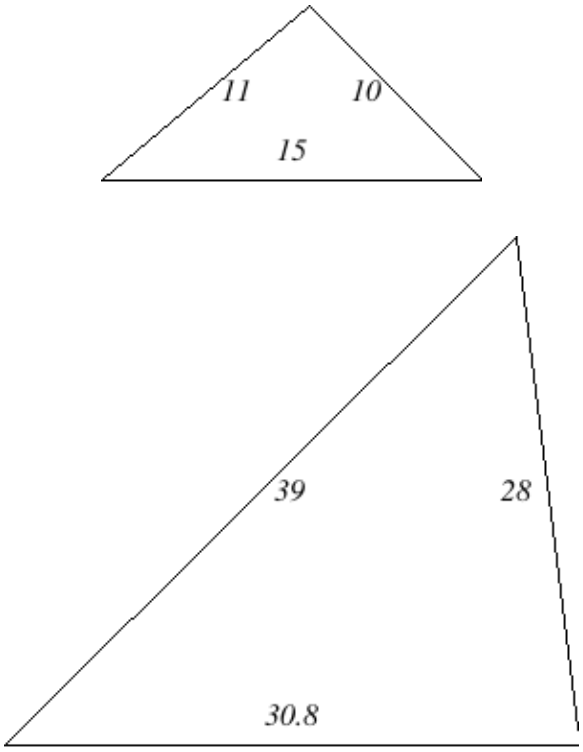


The first triangle has sides of lengths 15, 10, and 11. The second triangle has sides of lengths 30.8, 28, and 39.
Are the triangles above similar?

1. no
2. yes

The triangles are not drawn at the same scale.

16.



The first triangle has sides of lengths 15, 10, and 11. The second triangle has sides of lengths 30.8, 28, and 39.

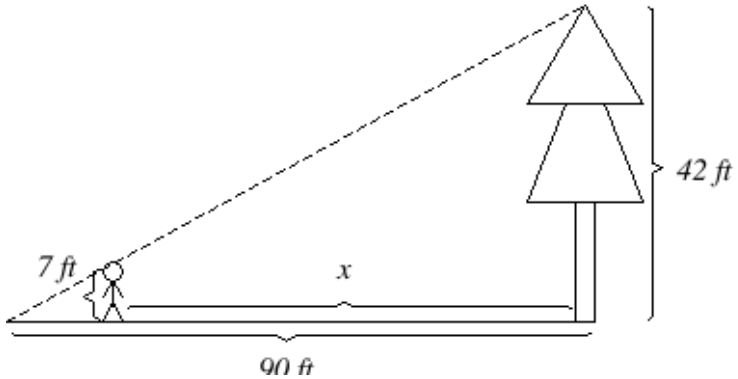
Are the triangles above similar?

1. no
2. yes

The triangles are not drawn at the same scale.

Similar Triangles in Context. Identify similar triangles in each application, then use the property of side ratios to calculate the requested value(s).

17. Suppose you are standing such that a 42-foot tree is directly between you and the sun. If you are 7 feet tall and the tree casts a 90-foot shadow, how far away from the tree can you stand and still be completely in the shadow of the tree?

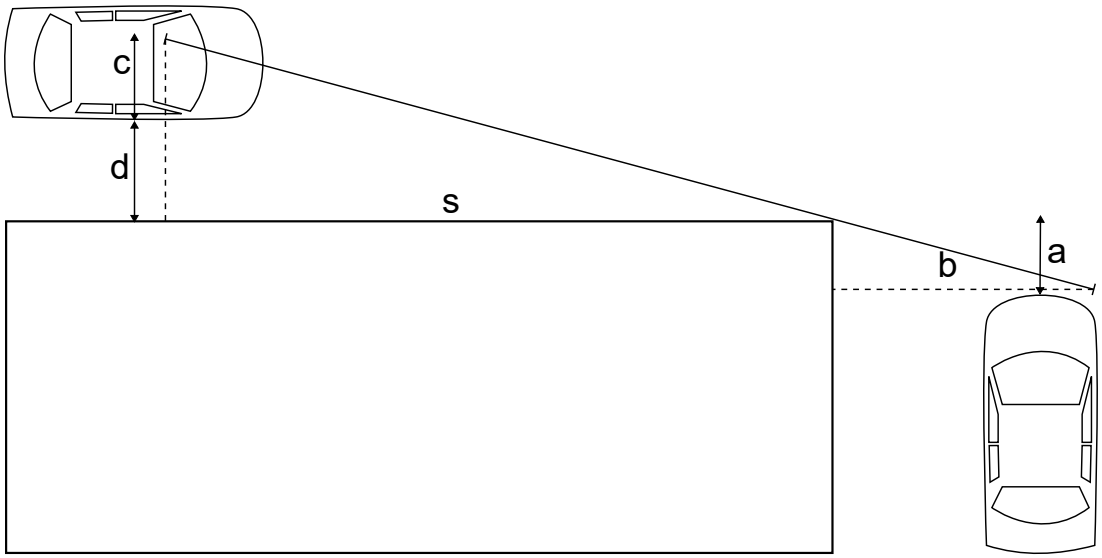


The distance between you and the tree is _____ ft (Round to 1 decimal place.)

18. A stick 1.0 meter long casts a shadow 0.70 meters long. A building casts a shadow 17.0 meters long. How tall is the building? Use the rules of working with significant figures to round.

_____ meters

19.



A building blocks the views of two drivers. The car driving from left to right (starting at the top left in the diagram) has just reached the location where its driver first sees the car waiting at the intersection to pull out heading up (starting at the bottom right in the diagram). Suppose the car waiting is 25.0 ft to the right of the building and 3.4 ft down from the intersection. Suppose as well that the driver of the moving car is 6.2 ft from the right side of the car, and the right side of the car is 4.0 ft from the building. How far is the driving car from the edge of the building (labeled s)? _____

Round to the nearest foot.

If the car is traveling to the right at 55 mph, how much time, in seconds, is there between first seeing the other car, and entering the intersection? _____

Round to the nearest second.

Quantitative Literacy. Proportions are needed to answer these more involved questions.

20. Aircraft flying at 18,000 ft MSL and above (called the flight levels) use an altimeter setting of 29.92 regardless of the local setting. Note higher pressure is lower in the atmosphere. 29.92 is higher pressure than 27.91, and occurs at a lower altitude.

If the local altimeter setting is 27.91 inHg, what is the difference in altimeter settings from 29.92 inHg? This should be a positive number. _____ inHg

In the lower atmosphere the pressure drops approximately 1 inHg per 1000 feet of elevation gained. Based on this, how many feet does the pressure difference (previous question) represent? _____ ft

Because the plane is using 29.92 inHg instead of 27.91 inHg, they are flying that many feet lower than their altimeter indicates. What is their actual altitude? _____ ft MSL

Because this actual altitude means they could conflict with the aircraft flying at 17,500, aircraft will not be allowed to fly at 18,000 ft MSL on this day.

Given how much lower they would be than indicated (as you calculated above), how much higher must they fly than 18,000 ft MSL? Note this must be a multiple of 1000 ft (FAA regulations), so round up to the nearest thousand. _____

21. If all know about two triangles is the lengths of the thee sides of each, in what order(s) should we try pairing the sides to test if they are similar? Suppose $\left|\overline{AB}\right| = 8$, $\left|\overline{BC}\right| = 12$, $\left|\overline{CA}\right| = 16$, $\left|\overline{DE}\right| = 24$, $\left|\overline{EF}\right| = 16$, and $\left|\overline{FD}\right| = 32$. To which sides of triangle ABC should each side of triangle DEF be

compared to test for similarity?

1. AB
2. BC
3. CA

1. EF
2. FD
3. DE

2.5 Medical Ratios

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Setup and solve proportions (skill)

This section uses medical applications, primarily determine medicine dosages, to illustrate the use of ratios including change of units and proportions. In each example look for how the ratio is recognized and how the information provided is used to setup a calculation.

A common application of ratios in medicine is creating drugs of a desired strength. For example some drugs need to be administered based on the body weight of the patient. This requires the medical personnel to mix the drug they have on hand to the needed strength.

We will work the following three types of medical problems.

- Measure drug concentration
- Dilute a drug to a lower concentration
- Determine how much drug to use

2.5.1 Rounding in Medical Applications

Rounding for medical applications will be dictated by two factors. First, if an amount is more precise than human and/or machine can measure, then that precision is of no value. For example, if the syringe pumps are capable of measuring a hundredth of a milliliter, then we round to a hundredth. Second, a hospital or other medical provider may have a policy.

Just as with significant digits, rounding will occur solely at the end of a calculation. Intermediate rounding could change a number. Rounding at the end is a practical necessity as noted above.

2.5.2 Terminology

This section defines terminology used in medicine and sciences about solution concentrations that we need for the ratio examples in this section.

The active ingredient in a drug is often added to a inactive ingredient (often liquid) to administer it. This liquid is known as a **diluent**. The diluent might be water, saline solution, or other substances.

The substance (active ingredient for medicine) which may be a powder or another liquid to which the diluent is added is called the **solute**. The solute is disolved in the diluent. For example salt (solute) is disolved in water (diluent) to make saline solution.

Even if the drug can be administered directly (e.g., is already liquid) we sometimes need to dilute the **stock solution** (undiluted drug) for ease of use.

In some problems the drug mixture will be divided into parts. These parts are sometimes called **aliquots**. For example when testing substances (like blood samples) we may divide the sample drawn into multiple aliquots, one for each test to be run.

The most important concept is measuring how concentrated a solution is. This enables providing sufficient and safe amounts of drugs. There are three common ways concentration is written. These three are examples of how ratios can present the relationship between quantities in different ways. Being able to change between the different presentations of concentration will demonstrate your ability to understand and use ratios accurately.

Definition 2.5.1 Dilution Ratio. The **Dilution Ratio** is the ratio of solute (drug) to diluent.

If the solute is a liquid, then this is in units of volume per volume (e.g., mL per mL). For example a dilution ratio of 1:4 means 1 mL of drug to 4 mL of diluent giving 5 mL of solution.

This expression of concentration is unlikely to be used for dry solutes. ◆

Definition 2.5.2 Percent Concentration. The **Percent Concentration** is the ratio of mass of solute (drug) to 100 mL of diluent.

If the solute is liquid, then this is in units of volume per volume. For example if the are 2 mL of drug per 100 mL of solution, then the percent concentration is 2/100 or 2%.

If the solute is solid (e.g, powder) then this is in units of mass per volume. For example, if there are 2 mg of drug per 100 mL of solution, then the percent concentration is 2/100 or 2%. Note this is neither percent by volume nor percent by mass as would be used in science. ◆

These examples illustrate the meaning of these terms.

Example 2.5.3 A solution is produced from 3 mL of concentrated chloroform and 37 mL of water.

(a) What is the dilution ratio?

The dilution ratio is the ratio of the solute to the diluent. We are given both. The dilution ratio is 3/37.

(b) Calculate the percent concentration.

The percent concentration is the ratio of the solute to the total solution by volume written as a percent. We are given the volume of the solute (3 mL) and have calculated the total volume in the previous task (40 mL). Thus the percent concentration is $\frac{3}{40} = 0.075$ which is 7.5%.

▲

Example 2.5.4 Saline solution consists of the solute salt (sodium chloride) dissolved in the diluent pure water. The saline solution most commonly used in medical applications is 9 g of the solute salt, which is a solid, dissolved in enough water to make 1 liter of solution. Because this is a solid disolved in a liquid (which does not meaningfully increase the volume) percent concentration makes the most sense for expressing the concentration.

To calculate the percent concentration we need to express the volume in milliliters, because percent concentration is the ratio of solute to 100 milliliters. We will convert units using Table 1.1.9. The percent concentration is

$$\begin{aligned} \frac{9 \text{ g}}{1000 \text{ mL}} &= \\ \frac{9 \text{ g}}{1000 \text{ mL}} \cdot \frac{1/10}{1/10} &= \\ \frac{0.9 \text{ g}}{100 \text{ mL}} &= 0.009 \end{aligned}$$

or 0.9%.

Note how comparing the information provided to the definition showed us we needed to perform a unit conversion. The definitions also stated which number is numerator and denominator and what form to use for a final expression (e.g., percent or fraction). ▲

Example 2.5.5 Clorox® Disinfecting Bleach contains 7.0% sodium hypochlorite in water. Sodium hypochlorite is a liquid. This means the percent concentration is 7.0%. One commercially available size contains 11 oz. What is the dilution ratio?

Looking at the defintion of dilution ratio, we see that we need to know the amount of solute (sodium hypochlorite) in the 11 oz. Because we are given a percent concentration, we can calculate this amount using the technique of Example 2.1.5. There is $11 \text{ oz} \cdot 0.070 = 0.77$ oz of sodium hypochlorite.

For the dilution ratio we need the amount of water added. Because the total solution is 11 oz and 0.77 oz is bleach, the amount of water is $11 - 0.77 = 10.23$ oz. Thus the dilution ratio is $\frac{0.77}{10.23}$.

This ratio is hard to interpret. We could express it with the numerator as one to indicate how much diluent is added to one part solute. We can calculate this multiple ways. The first way is to scale the ratio so the numerator is one.

$$\begin{aligned}\frac{0.77}{10.23} &= \frac{0.77}{10.23} \cdot \frac{1/0.77}{1/0.77} \\ &= \frac{1}{10.23/0.77} \\ &\approx \frac{1}{13.29}.\end{aligned}$$

Another option is to write this as a proportion.

$$\begin{aligned}\frac{0.77}{10.23} &= \frac{1}{R}. \\ \frac{10.23}{0.77} &= \frac{R}{1}. \\ \frac{10.23}{0.77} &= R. \\ 13.28571429 &\approx R. \\ 13.29 &\approx R.\end{aligned}$$

The ratio $\frac{1}{13.29}$ means there is one ounce of bleach for every 13.29 ounces of water. ▲

Note, the percent concentration for a pure substance is 100%.



Standalone
In Context

Use these Checkpoints to test your ability to calculate these ratios.

2.5.3 Dilution

This section shows how to use knowledge of proportions to perform calculations required in medicine. Dilution ratios or factors tell us a desired ratio, and we know the initial ratio. This pair allows us to setup a proportion.

This example shows us how to apply a dilution ratio (dilute our solution).

Example 2.5.6 A doctor orders 120 mL of 50% solution of Ensure every two hours. How much Ensure (liquid) and water is needed?

50% is a percent concentration. This means the Ensure should be 50% of the total volume (120 mL). $120 \cdot 0.50 = 60$ mL of Ensure. This leaves $120 - 60 = 60$ mL of water (diluent). ▲

In the example above the dilution ratio is 1/1, because there is the same volume of solute (Ensure) and diluent (water).

Working on dilutions is a proportion problem. This next example presents a scenario where we work a dilution problem backwards. Notice that the setup is still a proportion, and the solving is still just the algebra steps needed.

Example 2.5.7 One usage of dilution is to reduce the concentration so that instruments can accurately measure it. Consider trying to measure an acid without dissolving the tools used to measure it.

A sample of a suspected high blood glucose value was obtained. According to the manufacturer of the instrument used to read blood glucose values, the highest glucose result which can be obtained on this particular instrument is 500 mg/dL. When the sample was run, the machine gave an error message (concentration too high).

The serum was then diluted to 1/10 and retested. The machine gave a result of 70 mg/dL. What was the initial concentration?

Note that the ratio is milligrams to decilitres (weight to volume). In these types of problems the amount of substance is so small that it does not affect the volume.

Before we jump into an equation, let's try an experiment. That's right, in math we do not have to know what to do when we start. We will try something, learn from it, and maybe revise our approach based on what we learned.

This is a dilution problem which means we can setup the proportion

$$\frac{\text{solute mg}}{\text{diluent dL}} = \frac{70 \text{ mg}}{\text{dL}}.$$

We are trying to determine the amount of blood sugar in the sample, so the solute portion is unknown. We also do not have the size of sample taken. We will experiment to see how this affects the problem.

Suppose we take 1 dL of the original serum. Because the blood sample is so small, we can calculate as if all the volume is the diluent. That is we started with 1 dL and added more to dilute. To dilute to a ratio of 1/10 we need to add $10 - 1 = 9$ dL of diluent. No blood glucose was added thus the concentration is changed only by the diluent. Thus the concentration proportion is now

$$\begin{aligned} \frac{C + 0 \text{ mg}}{1 + 9 \text{ dL}} &= \frac{70 \text{ mg}}{\text{dL}} \\ \frac{C \text{ mg}}{10 \text{ dL}} &= \frac{70 \text{ mg}}{\text{dL}} \\ C \text{ mg} &= \frac{70 \text{ mg}}{\text{dL}}(10 \text{ dL}) \\ C &= 700 \text{ mg}. \end{aligned}$$

Did this result depend on our selecting 1 dL of the original serum? If we are uncertain we can try the problem again and select 2 dL of the original serum. To figure out the total amount of which 2 is 1/10, we can treat this like Example 2.3.7

$$\begin{aligned} \frac{1}{10} &= \frac{1}{10}\frac{2}{2} \\ &= \frac{2}{20}. \end{aligned}$$

This means we need $20 - 2 = 18$ dL of diluent to have the desired dilution ratio. Also we will have twice as much of the blood glucose.

$$\begin{aligned}
\frac{2C + 0 \text{ mg}}{2 + 18 \text{ dL}} &= \frac{70 \text{ mg}}{\text{dL}} \\
\frac{2C \text{ mg}}{20 \text{ dL}} &= \frac{70 \text{ mg}}{\text{dL}} \\
2C \text{ mg} &= \frac{70 \text{ mg}}{\text{dL}}(20 \text{ dL}) \\
2C \text{ mg} &= 1400 \text{ mg} \\
\frac{2C \text{ mg}}{2} &= \frac{1400 \text{ mg}}{2} \\
C &= 700 \text{ mg}
\end{aligned}$$

The result is the same. This makes sense, because we are setting up a proportion, and ratios do not depend on the amount.

We can be confident that the original serum sample had a blood glucose level of 700 mg/dL. ▲

Sometimes we dilute more than one time. Here we experiment to determine what the effect of **serial dilution** is upon the resulting dilution ratio.

Example 2.5.8 Suppose you have a solution consisting of 10 mL of acyl chloride and 90 mL of water. If this is diluted to a dilution ratio of 1/2 and then diluted again to a dilution ratio of 1/3, what is the final dilution ratio?

Solution. We can do the calculations one at a time. First we calculate the original concentration.

$$\begin{aligned}
\frac{10 \text{ mL}}{10 + 90 \text{ mL}} &= \frac{10}{100} \\
&= \frac{1}{10}.
\end{aligned}$$

To dilute to a ratio of 1/2 we can calculate the amount of diluent to add as a proportion problem like in Example 2.4.1.

$$\begin{aligned}
\frac{1}{2} &= \frac{100 \text{ mL}}{T \text{ mL}} \\
\frac{2}{1} &= \frac{T \text{ mL}}{100 \text{ mL}} \\
200 \text{ mL} &= T.
\end{aligned}$$

The *total* will be 200 mL so we need to add $200 \text{ mL} - 100 \text{ mL} = 100$ mL of additional diluent. Note at this point the concentration is

$$\frac{10 \text{ mL acyl chloride}}{200 \text{ mL diluent}} = \frac{1}{20}.$$

To dilute again to a ratio of 1/3 we can calculate the amount of diluent to add

$$\begin{aligned}
\frac{1}{3} &= \frac{200 \text{ mL}}{T \text{ mL}} \\
1 \cdot (T \text{ mL}) &= 3 \cdot (200 \text{ mL}) \\
T &= 600 \text{ mL}.
\end{aligned}$$

The total will be 600 mL so we need to add $600 \text{ mL} - 200 \text{ mL} = 400$ mL of additional diluent. At this point the concentration is

$$\frac{10 \text{ mL acyl chloride}}{600 \text{ mL diluent}} = \frac{1}{60}.$$

Now we can determine what the resulting dilution ratio after diluting twice (1/2 and then 1/3).

$$\frac{1}{10} \cdot F = \frac{1}{60}$$

$$10 \cdot \frac{1}{10} \cdot F = 10 \cdot \frac{1}{60}.$$
$$F = \frac{1}{6}.$$

Notice that $\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$, that is, the resulting dilution ratio is the product of the serial dilutions. This relationship is always true for serial dilution. ▲

2.5.4 Dosage

If we know the concentration of a drug, we can determine how much is needed for a given dose. These are proportion problems that require change of units.

In medicine some substances are measured in **International Unit** or IU. For each substance this is defined by the effect of that amount of the drug.

Example 2.5.9 One IU of insulin is 0.0347 mg. A common concentration of insulin is U-100 which is 100 IU/mL. This is produced by combining 100 units of insulin in one mL of diluent.

If a person needs 2 units of insulin, how many mL of solution will that be?

This can be solved as a proportion because we are asked for an amount that matches a ratio (concentration). The concentration is 100 IU/mL. Because we are solving for a number of mL, we write the proportion as

$$\frac{\text{mL}}{100 \text{ IU}} = \frac{v \text{ mL}}{2 \text{ IU}}$$
$$\frac{\text{mL}}{100 \text{ IU}} \cdot (2 \text{ IU}) = \frac{v \text{ mL}}{2 \text{ IU}} \cdot (2 \text{ IU})$$
$$\frac{2 \text{ mL}}{100} = v \text{ mL}$$
$$\frac{1}{50} \text{ mL} = v.$$
$$0.02 \text{ mL} = v.$$

Because we have a ratio of desired amount to provided amount we can also solve this problem as a percent.

The ratio is desired amount per provided amount. In this case $\frac{2 \text{ IU}}{100 \text{ IU}} = \frac{1}{50} = 0.02$ which is 2%. We therefore want 2% of the 1 mL (from 100 IU/1 mL) or 0.02 mL. ▲

Example 2.5.10 A label reads "2.5 mL of solution for injection contains 1000 mg of streptomycin sulfate." How many milliliters are needed to contain 800 mg of streptomycin?

Solution 1. Because a ratio is given (1000 mg/2.5 mL) and we want to scale this down (to 800 mg), we can set this up as a proportion.

We want to solve for volume (mL), so we set up the proportion as follows.

$$\frac{2.5 \text{ mL}}{1000 \text{ mg}} = \frac{v \text{ mL}}{800 \text{ mg}}.$$
$$\frac{2.5 \text{ mL}}{1000 \text{ mg}} \cdot (800 \text{ mg}) = \frac{v \text{ mL}}{800 \text{ mg}} \cdot (800 \text{ mg}).$$
$$2 \text{ mL} = v.$$

Solution 2. Because we have a ratio of desired amount to provided amount we can also solve this problem as a percent.

The ratio is desired amount/provided amount. In this case $\frac{800 \text{ IU}}{1000 \text{ IU}} = \frac{4}{5} = 0.8$ which is 80%. We therefore want 80% of the 2.5 mL dose or $0.8 \cdot 2.5 = 2$ mL. ▲



Standalone
In Context

Checkpoint 2.5.11 A physician ordered indomethacin 75 mg. indomethacin has a concentration of 25 mg per 5 milliliters.

What volume should be administered? ____ mL

A physician may prescribe a medicine and specify a total amount and a speed at which it should be delivered. For IV's this is called **drop factor** and is specified as a number of drops per minute. Medical personnel calculate how long to operate the IV so that the total amount of drug prescribed is delivered in the specified time.

Example 2.5.12 Give 1500 mL of saline solution IV with a drop factor of 10 drops per mL at a rate of 50 drops per minute to an adult patient. Determine how long in hours the IV should be administered.

The rate is specified in drops and the amount is specified in mL which means we need to convert units. This will be done like Example 1.1.20. The units suggest we should multiply the conversion ratios as follows.

$$\frac{50 \text{ drops}}{\text{minute}} \cdot \frac{\text{mL}}{10 \text{ drops}} = \frac{5 \text{ mL}}{\text{minute}}$$

Now that we know the rate in mL, we can set up a proportion so that time calculated per total grams of medication matches the specified rate. Notice how we invert the rate to make the algebra easier.

$$\begin{aligned} \frac{T \text{ min}}{1500 \text{ mL}} &= \frac{1 \text{ min}}{5 \text{ mL}}. \\ \frac{T \text{ min}}{1500 \text{ mL}} \cdot (1500 \text{ mL}) &= \frac{1 \text{ min}}{5 \text{ mL}} \cdot (1500 \text{ mL}). \\ T &= 300 \text{ min}. \end{aligned}$$

The final step is to convert minutes to hours. This is another unit conversion problem using a conversion from Table 1.1.2. The units suggest we can multiply the 300 minutes by the conversion ratio.

$$300 \text{ minutes} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = 5 \text{ hours}$$

▲



Standalone
In Context

Example 2.5.13 Amoxicillin is an antibiotic obtainable in a liquid suspension form, part medication and part water, and is frequently used to treat infections in infants. One formulation of the drug contains 125 mg of amoxicillin per 5 mL of liquid. A pediatrician orders 150 mg per day for a 4-month-old child with an ear infection. How much of the amoxicillin suspension would the parent need to administer to the infant in order to achieve the recommended daily dose?

Solution. Here we need to scale the amount (from 125 mg to 150 mg). This is a proportion problem, that is, the ratio of medicine to volume is the same so we can setup an equation based on the drug concentration.

$$\begin{aligned} \frac{125 \text{ mg}}{5 \text{ mL}} &= \frac{150 \text{ mg}}{A \text{ mL}} \\ \frac{5 \text{ mL}}{125 \text{ mg}} &= \frac{A \text{ mL}}{150 \text{ mg}} \\ \frac{5 \text{ mL}}{125 \text{ mg}} \cdot (150 \text{ mg}) &= A \text{ mL} \\ 6 \text{ mL} &= A. \end{aligned}$$

▲

Checkpoint 2.5.14 A 5% dextrose solution (D5W) contains 5 g of pure dextrose per 100 mL of solution. A doctor orders 500 mL of D5W IV for a patient. How much dextrose does the patient receive from that IV?

Example 2.5.15 A sample of chloroform water has a dilution ratio of 1:39. If 2 mL of chloroform are needed how many milliliters total are needed?

Solution. A dilution ratio of 1:39 indicates that 1 mL of chloroform is in $1 + 39 = 40$ mL total of solution. We can setup a proportion to answer this.

$$\begin{aligned} \frac{1 \text{ mL}}{40 \text{ mL}} &= \frac{2 \text{ mL}}{T \text{ mL}} \\ \frac{40 \text{ mL}}{1 \text{ mL}} &= \frac{T \text{ mL}}{2 \text{ mL}} \\ \frac{40 \text{ mL}}{1 \text{ mL}} \cdot 2 \text{ mL} &= T \text{ mL} \\ 80 \text{ mL}. \end{aligned}$$

▲

2.5.5 Exercises

1. A 1 litre (1,000 mL) IV bag of dextrose solution contains 40 g of dextrose. Find the ratio of grams per millilitre of dextrose. (Enter your answer in fraction form.) _____

2. Find the flow rate (in drops/min) for the given IV (assume a drop factor of 15 drops/mL).
1200 mL in 4.0 h
_____ drops/min

3. Find the length of time (in h) the IV should be administered (assume a drop factor of 10 drops/mL).
1,000 mL at a rate of 60 drops/min
_____ h

4. A label reads: "2.5 mL of solution for injection contains 1,000 mg of streptomycin sulfate." How many millilitres are needed to give 600 mg of streptomycin?
_____

5. Quinidine gluconate is a liquid mixture, part medicine and part water, which is administered intravenously. There are 60.0 mg of quinidine gluconate in each cubic centimeter (cc) of the liquid mixture. Dr. Martinz orders 204 mg of quinidine gluconate to be administered daily to a patient with malaria.
How much of the solution would have to be administered to achieve the recommended daily dosage?
_____ cc

6. Albuterol is a medicine used for treating asthma. It comes in an inhaler that contains 15 mg of albuterol mixed with a liquid. One actuation (inhalation) from the mouthpieces delivers a 90 μg dose of albuterol. (Reminder: 1 mg = 1000 μg .)
a.) Dr. Olson orders 2 inhalations 4 times per day. How many micrograms of albuterol does the patient inhale per day?
_____ μg
b.) How many actuations are contained in one inhaler?
_____ actuations
c.) Precious is going away for 7 months and wants to take enough albuterol to last for that time. Her physician has prescribed 2 inhalations 4 times per day. How many inhalers will Precious need to take with her for the 7 period? Assume 30-day months.
Hint: she can't bring a fraction of an inhaler, and she does not want to run out of medicine while away.
_____

7. Amoxicillin is a common antibiotic prescribed for children. It is a liquid suspension composed of part amoxicillin and part water.
In one formulation there are 175 mg of amoxicillin in 9 cubic centimeters (cc's) of the liquid suspension. Dr. Scarlotti prescribes 525 mg per day for a 2-yr old child with an ear infection.
How much of the amoxicillin liquid suspension would the child's parent need to administer in order to achieve the recommended daily dosage?
_____ cm^3

8. Diphenhydramine HCL is an antihistamine available in liquid form, part medication and part water. One formulation contains 15 mg of medication in 7 mL of liquid. An allergist orders 60-mg doses for a high school student. How many milliliters should be in each dose?
_____ mL

9. How many mL of sodium hydroxide are required to prepare 900 mL of a 15% solution? Assume the sodium hydroxide dissolves in the solution and does not contribute to the overall volume.
_____ mL

Dilution Ratios. Answer these questions involving dilution ratios.

10. You are asked to make a 1/9 dilution using 8 mL of serum. How much diluent do you need to use?
_____ mL

11. A clinical lab technician determines that a minimum of 65 mL of working reagent is needed for a procedure. To prepare a $\frac{1}{7}$ dilution of the reagent from a stock solution, one should measure 65 mL of the reagent and _____ mL of the diluent.

12. A patient's glucose result is suspected to be outside the range of the analyzer, so the techs decide to dilute the sample before running it. 40 microliters of serum is added to 120 microliters of diluent and the diluted sample is analyzed. The analyzer reads that the glucose value of the diluted sample is 31 $\frac{mg}{dL}$.

What was the ratio the sample was diluted to?

_____

What is the glucose value of the original sample?

_____ $\frac{mg}{dL}$

13. A thyroid peroxidase antibody test was performed on a 45 year old man. The dilution sequence was 15 μL serum added to 120 μL of diluent in tube 1. Then 60 μL from tube 1 was added to 240 μL of diluent in tube 2. Finally 45 μL from tube 2 was added to 225 μL of diluent in tube 3.

All dilution factors should be given as fractions.

a.) What is the dilution ratio in tube 1? Your answer should be in the form of $\frac{1}{\text{number}}$.

_____

b.) What is the ratio of the liquid from tube 1 in tube 2 after they are mixed?

_____

c.) What is the ratio of the liquid from tube 2 in tube 3 after they are mixed?

_____

d.) What is the ratio of the original serum in tube 3? That is, what is the overall (serial) dilution ratio?

_____

2.6 Project: False Position

Project 2.6.1 Method of False Position. In this project, we are going to learn about an ancient algebraic technique that is built around correcting guesses. We may gain greater appreciation for the value of *wrong* guesses and what we can gain from them.

(a) Solve the following equation any way you would like.

$$x\left(1+\frac{1}{3}+\frac{1}{4}\right)=14.$$

Check your answer using technology.

(b) Notice that 12 is the least common multiple of 3 and 4: the denominators. Distribute 12 in the following expression.

$$12\left(1+\frac{1}{3}+\frac{1}{4}\right).$$

Is this bigger, equal to, or smaller than 14?

(c) We multiplied by a convenient number, which is not quite right. Because it is multiplication we can scale (multiply) our not quite right guess to make it right. Consider

$$y\cdot 12\left(1+\frac{1}{3}+\frac{1}{4}\right)=14.$$

Replace 12 times the sum with your result from the previous step.

Solve the resulting equation for y.

(d) Note that y is the correction to our guess of 12. Calculate $y \cdot 12$.

This will match your original solution. If not, check your calculations.

(e) This method is called *false position* because it guesses a convenient number which is typically false then corrects it. One of the original motivations for this method was the lack of a useful notation for fractions (it dates to the Sumerians and ancient Egyptians).

Many people today use similar methods when dealing with fractions. What is a reason people might distribute a convenient number before doing the solving?

2.7 Project: Arclength Estimation

Project 2.7.1 Estimating Arc Lengths. In aviation it is sometimes useful to estimate a distance between points as the length of a circular arc. This results from navigation methods (search for VOR and DME arc if curious). To estimate on the fly they use what is known as the 60:1:1 approximation. It means that 60 miles from a point a one degree arc is approximately one mile in length. In aviation the distances would be in nautical miles (nm), but the ratio does not change if we use statute miles (the usual type).

Here we will practice using the method to approximate then check why it works.

(a) *Using the Ratio.*

(i) View Example 2.7.2 to Example 2.7.4.

(ii) What is the arclength of 2 degrees at a distance of 30 miles?

(iii) What is the arclength of 5 degrees at a distance of 30 miles?

(iv) What is the arclength of 10 degrees at a distance of 20 miles?

(b) *Explaining the Ratio.*

In four steps we will figure out the length of a one degree arc on a circle with radius 60 miles. Use $P = 2\pi r$ where P is the perimeter and r is the radius.

(i) Calculate the perimeter of a circle with radius 60 miles using the formula

(ii) Calculate the perimeter of a semi-circle (half circle) with radius 60 miles.

(iii) Calculate the perimeter of a quarter of a circle with radius 60 miles.

(iv) Calculate the perimeter of 1/360 of a circle with radius 60 miles.

(c) Next we will consider how accurate this estimate is.

(i) Compare the result of the arclength of 1/360 of a circle from the previous step to the estimation of 1 miles from the 60:1:1. What is the error using 3 decimal places from your calculation above?

(ii) Calculate the arclength for a 15° arc of a circle with radius 20 miles using $P = 2\pi R$ and using the 60:1:1 approximation. What is the difference between these calculations using 3 decimal places?

(iii) At 90 mph how long does it take to travel the difference in distance from the previous step?

Example 2.7.2 Calculate the arclength of 3 degrees at 60 miles.

Solution. If each degree is one mile then 3 degrees is $A = 3 \cdot 1 = 3$ miles. ▲

Example 2.7.3 Calculate the arclength of 1 degree at 30 miles.

Solution. At 30 miles we are only half way ($30/60 = 1/2$), so the length is $A = \frac{1}{2} \cdot 1 = \frac{1}{2}$ miles. ▲

Example 2.7.4 Calculate the arclength of 4 degree at 18 miles.

Solution. The radius is $(18/60 = 3/10)$ of the usual. Thus each degree is $\frac{3}{10}$ of a mile. This arc is 4° so the length is $A = \frac{3}{10} \cdot 4 = 1.2$ miles. ▲

Chapter 3

Models

3.1 Linear Expressions

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)
- Read and interpret models (critical thinking)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

Section 1.3 presented models in general. This section presents linear models. First, we look at some examples and learn how the pieces of a linear model work. Next, we learn to write linear models given a description of a problem. After that we practice solving for different parts of a linear equation. Section 3.3 will introduce a more in depth look at identifying linear models.

3.1.1 Linear Models

This section presents examples of linear models and provides an explanation for the two parts of a linear model.

A linear model (equation) can be written in the following, equivalent forms.

- $y = mx + b$
- $Ay + Bx + C = 0$

The second form can be solved for y which will make it look like the first form. The models in this section will be in the first form which you may know as slope intercept. The m is commonly described as the **slope** which is a measure of how steep the line is. In these models we will generalize slope to a rate. The b is commonly called the **intercept** which indicates a **shift** of the line up or down. This shift will have a meaning in each application.

Model 3.1.1 Temperature Change with Altitude. *As a result of atmospheric physics, temperature decreases as the distance above the ground increases. For lower altitudes this can be modeled as*

$$T_A = T_G - \left(\frac{3.5^\circ\ F}{1000\ ft}\right) A.$$

where

- *T_A is the expected temperature in degrees Fahrenheit at the specified altitude.*
- *T_G is the temperature at ground level in degrees Fahrenheit.*
- *A is the specified altitude in number of feet above ground level.*
- *$\frac{3.5^\circ}{1000\ ft}$ is the rate of temperature decrease.*

All temperatures are in Fahrenheit.

Before we use this model, we need to figure out what each part is. T_G is a parameter that we need to find before we use the model. A parameter is not a variable, rather it is a value (number) that we obtain from the circumstances and write into the model (equation) before we do any work.

In contrast the ratio $-\frac{3.5^\circ}{1000\text{ ft}}$ is a constant (not a parameter), because it is a result of atmospheric physics. This ratio is always in the equation, because it is not dependent on the location for this simplified model.

Temperature (T_A) and altitude (A) are variables which implies that the model shows a relationship between these two properties.

The model subtracting from the starting temperature results in a decrease of temperature from T_G. This implies that temperature decreases with altitude.

Every linear model (equation) has a rate. In this case $m = -\frac{3.5}{1000}$.

Every linear model has a shift. In this case $b = T_G$.

Now that we have analyzed the model, we are prepared to use it.

Example 3.1.2 If the temperature at ground level is 43° what is the temperature 1000 ft above ground level (AGL)? 2000 ft AGL, 3000 ft AGL, 3500 ft AGL?

Because fractions of a degree are not useful in making decisions like what clothes to wear, we will round to units.

Note $T_G = 43^\circ$. We need to calculate T_A for $A = 1000, 2000, 3000, 3500$.

$$\begin{aligned}
T_{1000} &= 43^\circ - \frac{3.5^\circ}{1000\text{ ft}}(1000\text{ ft}) \\
&= 39.5 \\
&= 43^\circ - 3.5^\circ \\
&= 39.5 \\
&\approx 40.
\end{aligned}$$

$$\begin{aligned}
T_{2000} &= 43^\circ - \frac{3.5^\circ}{1000\text{ ft}}(2000\text{ ft}) \\
&= 43^\circ - 7.0^\circ \\
&= 36.
\end{aligned}$$

$$\begin{aligned}
T_{3000} &= 43^\circ - \frac{3.5^\circ}{1000\text{ ft}}(3000\text{ ft}) \\
&= 43^\circ - 10.5^\circ \\
&= 32.5 \\
&\approx 33.
\end{aligned}$$

$$\begin{aligned}
T_{3500} &= 43^\circ - \frac{3.5^\circ}{1000\text{ ft}}(3500\text{ ft}) \\
&= 43^\circ - 12.25^\circ \\
&= 30.75 \\
&\approx 31.
\end{aligned}$$

Notice that we now know that it will be below freezing just above 3000 ft. ▲

An Equation for Time to Altitude. A fixed wing aircraft flown optimally climbs from a starting altitude at a fixed climb rate.

$$A_t = A_G + C \cdot t.$$

- A_t is the altitude after t minutes.
- A_G is the starting altitude (likely ground level) in feet mean sea level (MSL).
- C is the rate of climb in feet per minute.
- t is the time since the climb began in minutes.

Before we can use this equation, let's analyze it. We will need to know the parameters A_G and C. A parameter is not a variable, rather it is a value (number) that we obtain from the circumstances and write into the model (equation) before we do any work. A_G varies by airport, because they are at different altitudes. The rate C must be obtained for each plane and is often available in the aircraft's Pilot's Operating Handbook (POH).

Final altitude (A_t) and time (t) are the variables which implies that the model shows a relationship between time climbing and how high the plane is.

In this model everything is added which matches the increase of elevation over time (adding makes the altitude bigger).

Every linear model (equation) has a rate (the slope m in the equation). In this case $m = \frac{C \text{ ft}}{1 \text{ min}}$. Every linear model has a shift (the b in the equation), which may be zero. In this case $b = A_G$. $b = A_G = 0$ is possible because an aircraft can take off from sea level (e.g., float planes). This shift makes sense, because the climb starts at the altitude of the ground: the plane was already shifted up by being at that airport.

Now that we have analyzed this equation, we are prepared to use it.

Example 3.1.3 If a plane begins at 160 ft MSL and is climbing at 700 ft/min, how high will it be after 5 minutes? 10 minutes? 15 minutes?

These calculations are made as part of safety planning. The data is sufficiently accurate that rounding is not necessary. Rather we make conservative estimates of the parameters, so that there is always a safety buffer. In this case a conservative estimate for A_G is to round down: this will give us a lower altitude. If that lower altitude is safe, then altitude 5 feet higher will be safe as well. For the climb rate a conservative estimate is to round down as well. If we can reach an altitude climbing at 700 ft/min, then if we climb at 720 ft/min we will reach that safe altitude a little sooner.

Note

$$A_t = 160 \text{ ft} + \frac{700 \text{ ft}}{\text{min}} \cdot t.$$

The expected altitudes are

$$\begin{aligned} A_5 &= 160 \text{ ft} + \frac{700 \text{ ft}}{\text{min}} \cdot 5 \text{ min} \\ &= 160 \text{ ft} + 3500 \text{ ft} \\ &= 3660. \\ A_{10} &= 160 \text{ ft} + \frac{700 \text{ ft}}{\text{min}} \cdot 10 \text{ min} \\ &= 160 \text{ ft} + 7000 \text{ ft} \\ &= 7160. \\ A_{15} &= 160 \text{ ft} + \frac{700 \text{ ft}}{\text{min}} \cdot 15 \text{ min} \\ &= 160 \text{ ft} + 10500 \text{ ft} \\ &= 10660. \end{aligned}$$

Based on these calculations the plane will climb above 4500 ft MSL in between 5 and 10 minutes (closer to 5). ▲

Equation for Fuel Remaining. When operated at a fixed power setting a vehicle burns the same amount of gas per hour (or other time unit). This leads to the linear model

$$F_t = F_I - r \cdot t.$$

- F_t is the amount of fuel remaining after t minutes.
- F_I is the amount of fuel at the beginning.
- r is the rate (volume per time) at which fuel is being consumed.
- t is the time the vehicle has been operated.

Fuel amounts will be measured in units of volume like gallons or liters. Time will be measured in minutes or hours. The rate r is then in units such as gallons/hour or liters/min.

Before we use this model, let's analyze it. We will need to know the parameters F_I and r. These parameters are not variables (they remains the same the whole time the model is in use), rather they are values (numbers) that we obtain from the circumstances and write into the model (equation) before we do any work.

The initial fuel F_I is obtained by checking the fuel tanks or fuel gauges. The rate r is often is not shown during operation (fuel gauges show how much is remaining rather than how fast it is used). The rate can sometimes be obtained from vehicle documentation.

Final fuel (F_t) and time (t) are the variables which implies that the model shows a relationship between time flown and fuel available (left in the tanks).

Because fuel decreases the $r \cdot t$ term is subtracted decreasing the amount from F_I.

Every linear model (equation) has a rate. In this case $m = \frac{r \text{ gal}}{1 \text{ hr}}$ (or similar units).

Every linear model has a shift. In this case $b = F_I$. We can think of this shift in terms of the needle on the gas gauge moving up to represent the amount of fuel present.

Now that we have analyzed this equation, we are prepared to use it.

Example 3.1.4 If a car begins with 20 gallons of fuel and burns 1.55 gallons per hour, how much fuel will it have after 1 hour, 2 hours, 3 hours, 36 minutes?

A gallon is a large amount so we will maintain one decimal place precision. For safety we should always assume a larger fuel burn, so we will round fuel remaining down.

Solution. The model is

$$F_t = 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot t \text{ hr}.$$

Thus

$$\begin{aligned} F_1 &= 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot 1 \text{ hr} \\ &= 20 \text{ gal} - 1.55 \text{ gal} \\ &= 18.45 \\ &\approx 18.4. \end{aligned}$$

$$\begin{aligned} F_2 &= 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot 2 \text{ hrs} \\ &= 20 \text{ gal} - 3.10 \text{ gal} \\ &= 16.9. \end{aligned}$$

$$\begin{aligned} F_3 &= 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot 3 \text{ hrs} \\ &= 20 \text{ gal} - 4.65 \text{ gal} \\ &= 15.35 \\ &\approx 15.3. \end{aligned}$$

$$F_{0.6} = 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot 0.6 \text{ hrs}$$

$$= 20 \text{ gal} - 0.93 \text{ gal}$$
$$= 19.07$$
$$\approx 19.0.$$

▲

Use this Checkpoint to practice using a linear model.

Checkpoint 3.1.5 The expected temperature at a height above ground is given by
$T_A = T_G - \frac{3.5}{1000}A$
where T_A is the expected temperature in Fahrenheit
T_G is the temperature at ground level in Fahrenheit
A is the height above ground level in feet
If the temperature on the ground is 72°, what will it be at 2400 feet above ground level? ___
Answers should be rounded to the units place.

3.1.2 Building Linear Models

The previous section presented linear models, and illustrated using the models provided. This section presents problems that can be modeled as linear equations, and illustrates writing the model (equation) before using it.

A linear model has a starting point (shift, b) and rate (ratio, m). We need to identify these and then write the linear model

$$y = b + mx$$

with these values. We should also label units and explain any parameters.

We have used both $y = mx + b$ and $y = b + mx$ to describe a linear model. These are of course the same thing. In many applications it feels more appropriate to begin with the shift or offset, so we will prefer that version in this text.

Example 3.1.6 Consider rope that costs \$0.93 per foot with a shipping charge of \$7.64. We will produce a model for the cost of each purchase.

We can start by calculating the cost of some specific orders (experiment). Suppose we are purchasing 20 feet of this rope. The units (dollars/foot) suggest that we can multiply to calculate the cost. $20 \text{ ft} \cdot \frac{\$0.93}{\text{ft}}$.

Notice this multiplication is also the same as using a ratio (proportion). We could setup $\frac{\$0.93}{1\text{ft}} = \frac{C}{20 \text{ ft}}$ When we solve this we end up with the same multiplication $20 \text{ ft} \cdot \frac{\$0.93}{\text{ft}}$.

Next we must add the shipping charge. Thus the final cost is $20 \text{ ft} \cdot \frac{\$0.93}{\text{ft}} + \$7.64 = \$26.24$. Note there is no rounding because all numbers are exact and no fractions of a cent occurred.

Suppose we are purchasing 100 feet of this rope. The cost for the 100 feet will be

$$100 \text{ ft} \cdot \frac{\$0.93}{\text{ft}}.$$

Then we must add the shipping charge. Thus the final cost is $100 \text{ ft} \cdot \frac{\$0.93}{\text{ft}} + \$7.64 = \$100.64$.

We can now notice that this calculation is the same with any number of feet (unless the shipping charge increases for larger orders). So in general we can write the cost as

$$s \text{ ft} \cdot \frac{\$0.93}{\text{ft}} + \$7.64 = C.$$

This equation has a ratio (\$0.93/1 ft), which is the cost per foot, and also has a shift (+\$7.64), which is the fixed shipping cost. Thus this is another linear equation. ▲

When cost is set per linear foot, or per square yard, or similar per unit pricing we often end up with a linear model.

Example 3.1.7 At lower altitudes the barometric pressure typically drops 1 inHg for every 1000 feet of elevation gained (the air is less dense higher up). We will produce a model for pressure at altitude.

As in the previous example, we will start by calculating some specific pressures (experiment). If the pressure on the ground is 29.76 inHg, what do we expect the pressure to be flying at 4500 ft above ground level?

The pressure drop is a ratio $\frac{1 \text{ inHg}}{1000 \text{ ft}}$. The units suggest we can multiply $4500 \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}} = 4.5$ inHg. This is the drop in pressure. To calculate the resulting pressure we need $29.76 \text{ inHg} - 4.5 \text{ inHg} = 25.26$ inHg. We retain 2 decimal places because that is the traditional amount for reporting by meteorologists. Written as one calculation this is $T = 29.76 \text{ inHg} - \left(4500 \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}}\right)$.

If the pressure on the ground is 30.02 inHg what do we expect the pressure to be flying at 6000 ft above ground level?

The pressure drop is a ratio $\frac{1 \text{ inHg}}{1000 \text{ ft}}$. The units suggest we can multiply $6000 \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}} = 6$ inHg. This is the drop in pressure. To calculate the resulting pressure we need $30.02 \text{ inHg} - 6 \text{ inHg} = 24.02$ inHg. Written as one calculation this is $P = 30.02 \text{ inHg} - \left(6000 \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}}\right)$.

Notice we could do this same calculation for any altitude. So in general we can write

$$P_A = P_G - \left(A \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}}\right).$$

P_A is the pressure at the specified altitude. P_G is the pressure at ground level. A is the altitude above ground level. This is a linear equation with a ratio of (-1/1000) which is the drop in pressure with altitude, and a shift of P_G, which is the pressure on the ground. ▲

Example 3.1.8 We will construct a model (equation) that converts temperature in Fahrenheit to temperature in Celsius. Every 9 degrees Fahrenheit is only 5 degrees Celsius, so to convert between them, we must scale the degrees. Also they use different values for the starting point (which is the freezing point of water). Fahrenheit starts at 32° and Celsius starts at 0°. Notice we have a ratio and a shift, so this already looks like a linear model.

To begin with we will convert 52° F to Celsius. We will round to units, because this is just an example (no one will be injured in the demonstration of this model).

The first step is to determine how many degrees above freezing. Because 32° F is the freezing temperature of water, 52° F is $52° - 32° = 20°$ above freezing.

The next step is to scale the degrees. The conversion ratio is $\frac{9° \text{ F}}{5° \text{ C}}$. Converting the 20° above freezing is now a unit conversion. The units suggest we can multiply $20° \text{ F} \cdot \frac{5° \text{ C}}{9° \text{ F}} \approx 11° \text{ C}$. Notice we flipped the conversion ratio so the Fahrenheit degrees would divide to one. The result is 11° C above the freezing point of water in Celsius.

Finally we can add the degrees Celsius to the starting point (freezing temperature of water). Because that is 0°, we have $0° + 11° = 11°$ C.

If we write all of that as one step we obtain

$$(52° \text{ F} - 32° \text{ F})\frac{5° \text{ C}}{9° \text{ F}} + 0° \text{ C} = 11° \text{ C}.$$

Notice we could do this with any temperature. So if we call the temperature to convert T we have

$$C = (T - 32)\frac{5}{9} + 0.$$

This may not look like a starting point plus a ratio scaled. If we expand the expression we obtain

$$C = (T - 32)\frac{5}{9} + 0 = \frac{5}{9}T - \frac{160}{9}.$$

So this is a linear model. We prefer the first form of the equation because the numbers have meaning (e.g., 32° is the freezing point of water as opposed to 160/9 which has no useful meaning. ▲

The temperature conversion example illustrates an idea about models. We describe a linear model as having a ratio and a (that is one) shift. However, in the temperature conversion example there is a shift a ratio and another shift, or two shifts. We showed these can be combined as one shift. That is always true.

However, sometimes the version with multiple shifts is easier to understand. This will be true when we look at graphs of quadratics and exponentials (other forms of models).

3.1.3 Solving Linear Equations

The first section demonstrated using linear models to calculate values by plugging in the inputs. However, sometimes we know the result and want to calculate the input. This requires solving the linear equation. This section reviews solving linear equation starting with non-contextualized examples and then using some of the models presented above.

Before reading farther solve the equation $5x - 7 = 12$. What steps did you use? Why do they work? Example 3.1.9 is an example of solving another linear equation.

Example 3.1.9 Solve $-8x - 3 = 5$.

$$\begin{aligned} -8x - 3 &= 5. \\ -8x - 3 + 3 &= 5 + 3. \\ -8x &= 8. \\ \frac{-8x}{-8} &= \frac{8}{-8}. \\ x &= -1. \end{aligned}$$

We added three because it eliminates the -3 (undoes subtraction of 3). We divided by negative eight because it eliminates the -8 (undoes the multiplication by -8). These steps isolate the variable for which we are solving. ▲

Checkpoint 3.1.10 What is the solution to $19 = 2x + 9$? ___

Some linear equations need one more technique. What would you need to solve $17 - 4y = 14 - y$? Below is an example of solving a similar linear equation.

Example 3.1.11 Solve $17 - 4y = 5 + 2y$.

$17 - 4y = 5 + 2y.$	
$-5 + 17 - 4y = -5 + 5 + 2y.$	Collect constants on one side.
$12 - 4y = 2y.$	
$12 - 4y + 4y = 2y + 4y.$	Collect the variable on the other side.
$12 = 2y + 4y.$	
$12 = (2 + 4)y.$	Factoring reduces to one instance of the variable.
$12 = 6y.$	
$\frac{12}{6} = \frac{6y}{6}.$	Arithmetic to isolate y
$2 = y.$	

Because there were multiple instances of the variable in the initial equation, we had to combine like terms (factor and add). ▲

Checkpoint 3.1.12 What is the solution to $8x + 4 = 2x + 16$? ___

Another linear equation is $\frac{x}{3} + \frac{x}{4} = \frac{7}{12}$. How would you solve it?

We can solve this the same as in Example 3.1.11 but because there are factions as coefficients we will use an additional technique.

Example 3.1.13 Solve $\frac{x}{5} + \frac{2x}{7} = \frac{34}{35}$

$$\begin{aligned}
\frac{x}{5}+\frac{2x}{7}&=\frac{34}{35}.\\
5\cdot\left(\frac{x}{5}+\frac{2x}{7}\right)&=5\cdot\frac{34}{35}. && \text{Eliminate a denominator}\\
\frac{5x}{5}+\frac{10x}{7}&=5\cdot\frac{34}{35}. && \text{Distribute}\\
x+\frac{10x}{7}&=\frac{34}{7}.\\
7\cdot\left(x+\frac{10x}{7}\right)&=7\cdot\frac{34}{7}. && \text{Eliminate the other denominator}\\
7x+\frac{7\cdot 10x}{7}&=7\cdot\frac{34}{7}. && \text{Distribute}\\
7x+10x&=34.\\
(7+10)x&=34. && \text{Factor}\\
17x&=34.\\
\frac{17x}{17}&=\frac{34}{17}.\\
x&=2.
\end{aligned}$$

This is referred to as clearing denominators. We are once again eliminating division by multiplying. Always remember to distribute. Note, we could multiply once if we figured out the correct number (it would be 35 in this case), but there are no prizes for doing this fast, so you can do this either way. ▲

Now that we have practiced solving linear equations, we can use this skill with the models.

Example 3.1.14 Given the temperature model in Temperature Change with Altitude and supposing the temperature at ground level is 65, determine at what altitude we expect the temperature to be freezing. Round to the tens position; our measurements and sensors are not more precise.

In this case the model is $T_A = 65^\circ - \frac{3.5^\circ}{1000\text{ ft}}A$. We know $T_A = 32^\circ$ and we want to calculate A, the altitude in feet.

$$\begin{aligned}
32^\circ &= 65^\circ - \frac{3.5^\circ}{1000\text{ ft}}A\\
-65^\circ + 32^\circ &= -65^\circ + 65^\circ - \frac{3.5^\circ}{1000\text{ ft}}A\\
-33^\circ &= -\frac{3.5^\circ}{1000\text{ ft}}A\\
-\frac{1000\text{ ft}}{3.5^\circ}(-33^\circ) &= -\frac{1000\text{ ft}}{3.5^\circ}\left(-\frac{3.5^\circ}{1000\text{ ft}}A\right)\\
9428.571428\text{ ft} &= A.\\
9430\text{ ft} &\approx A.
\end{aligned}$$

▲

Example 3.1.15 Given the time to altitude model in An Equation for Time to Altitude and supposing that we are climbing from 80 ft MSL to 5000 ft MSL with a climb rate of 700 ft/min, how long will it take to complete the climb?

In this case the model is $A_t = 80\text{ ft} + \frac{700\text{ ft}}{\text{min}}t$. We know $A_t = 5000$ ft, and we want to know the time t.

$$\begin{aligned}
5000\text{ ft} &= 80\text{ ft} + \frac{700\text{ ft}}{\text{min}}t.\\
-80\text{ ft} + 5000\text{ ft} &= -80\text{ ft} + 80\text{ ft} + \frac{700\text{ ft}}{\text{min}}t.
\end{aligned}$$

$$\begin{aligned} 4920 \text{ ft} &= \frac{700 \text{ ft}}{\text{min}} t. \\ \frac{\text{min}}{700 \text{ ft}} \cdot 4920 \text{ ft} &= \frac{\text{min}}{700 \text{ ft}} \cdot \frac{700 \text{ ft}}{\text{min}} t. \\ 7.028571429 \text{ min} &= t. \\ 8 \text{ min} &= t. \end{aligned}$$

We round up as a safety margin: it is better to assume we need 8 minutes and be higher, than to hope we can recognize 0.03 of a minute (not quite 2 seconds). ▲

Use this Checkpoint to try solving a linear model.

Checkpoint 3.1.16 How long can you fly if you start with 73 gallons of fuel, burn 14 gallons per hour, and want land with one hour worth of fuel remaining? ____

Round down to the nearest tenth for a conservative estimate.

3.1.4 Identifying Linear Expressions

All of the equations in this section are linear. What can we use to identify linear expressions or linear equations? Table 3.1.17 shows examples of linear expressions and non-linear expressions.

Table 3.1.17 Linear and Non-linear

Linear	Non-linear
$5x + 3$	$5x^2 - x + 3$
$y = 11 - \frac{7}{13}x$	$y = \frac{17}{x}$
$7x - 9y = 8$	$3 - 2xy = 12$

Some equations that may not appear to be linear can be solved using the same methods.

Example 3.1.18 Solve $\frac{11}{x} + 2 = \frac{18}{x} - 5$.

$$\begin{aligned} \frac{11}{x} + 2 &= \frac{18}{x} - 5. \\ x \cdot \left(\frac{11}{x} + 2\right) &= x \cdot \left(\frac{18}{x} - 5\right). && \text{Clearing the denominators} \\ \frac{11x}{x} + 2x &= \frac{18x}{x} - 5x. && \text{Distribute.} \\ 11 + 2x &= 18 - 5x. && \text{Now it appears linear.} \\ -11 + 11 + 2x &= -11 + 18 - 5x. \\ 2x &= 7 - 5x. \\ 2x + 5x &= 7 - 5x + 5x. \\ 7x &= 7. \\ \frac{7x}{7} &= \frac{7}{7}. \\ x &= 1. \end{aligned}$$

▲

In Section 3.3 we will learn to identify linear models from data.

3.1.5 Exercises

Identify Shift and Ratio. For each model identify the shift (y-intercept) and ratio (slope).

1. Temperature change with respect to altitude can be modeled as

$$T = 15.04^\circ \text{ C} - \frac{0.00649^\circ \text{ C}}{1 \text{ m}} h$$

where
T is the resulting temperature.
h is the altitude mean sea level.
This is a linear model.
What is the shift (y-intercept)? _____
What is the ratio (slope)? _____
What are the units for h?

1. meters per Celsius
2. Celsius per meter
3. Celsius
4. meters

What are the units for T?

1. Celsius per meter
2. Celsius
3. meters per Celsius
4. meters

2. Pressure change with respect to altitude can be modeled as

$$P = 1013.25 \text{ mbar} - \frac{11.11 \text{ mbar}}{100 \text{ m}} h$$

where
P is the resulting atmospheric pressure.
h is the altitude mean sea level.
This is a linear model.
What is the shift (y-intercept)? _____
What is the ratio (slope)? _____
What are the units for h?

1. mbar
2. meters per mbar
3. meters
4. mbar per meter

What are the units for P?

1. meters
2. mbar per meter
3. mbar
4. meters per mbar

Linear expressions in applications. Answer these questions using the linear model described.

3. In certain deep parts of oceans, the pressure of sea water, P, in pounds per square foot, at a depth of d feet below the surface, is given by the following equation:

$P = 16 + \frac{4d}{11}$

If a scientific team uses special equipment to measures the pressure under water and finds it to be 340 pounds per square foot, at what depth is the team making their measurements?

Answer: The team is measuring at _____ feet below the surface.

4. You are on an oceanographic research expedition that began in San Juan, Puerto Rico on September 14.

The ship left port at 0630 hr on 14 September and covered a distance of 1835 km to the first drill location (Site 1) where you are going retrieve a drill core of seafloor sediments. The ship arrived at the first drill site at 1800 hr on 16 September.

Calculate the rate of travel (i.e., speed) of the ship during its transit to the first drill site. *Round your answer to the nearest tenth.*

_____ km/hr

5. Suppose you start with a full tank of gas (19 gallons) in your truck. After driving 5 hours, you now have 6 gallons left.

If x is the number of hours you have been driving, then y is the number of gallons left in the tank.

At what rate is the gas left in the tank changing? State your answer as a reduced fraction.

_ gallons per hour

Find an equation of a line in the form y = mx + b that describes the amount of gas in your tank.

$y =$ _____

6. A subscription plan for a GPS tracking device costs \$11.95 per month plus \$0.10 per location ping (sent once every 10 minutes). What is the monthly charge if the device was used for 3 hours and 7 minutes? _____

7. There are two options for a subscription plan for a GPS tracking device. One costs \$14.99 per month plus \$0.10 per location ping (sent once every 10 minutes). The other costs \$24.95 per month for unlimited location pings. What is the smallest number of minutes per month for which the second plan with fixed monthly cost is less expensive? ____

8. Vasya's initial salary is \$64200. If she receives a raise of \$4700 each year, what is the first year her salary is greater (or equal to) \$100,000? _____

9. Your friend Pat says to you, "I'm on a new diet plan and I've been able to lose about a pound a week." Come up with a linear equation that models this situation and use it to answer the following questions:

If Pat currently weighs 225 pounds, how much would Pat weigh in a year (52 weeks)?

_____ pounds

Assuming Pat stays on this plan and the equation is still valid, how much would Pat weigh in 4 years?

_____ pounds

Does this equation still seem like it would be valid after 4 years? Why or why not?

Solving in Models. Solve each equation for the variable indicated. Treat all other variables as parameters.

10. Solve the following formula for m

$c = amt$

$m =$ _____

Enter your answer as an expression.

But be careful...to enter an expression like $\frac{a+b}{3+m}$ you need to type *(a+b)/(3+m)*. You need parentheses for both the numerator and denominator.

11. Solve the formula $d = rt$ for t.

$t =$ _____

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

12. Solve the formula $V = \pi r^2 h$ for h. HINT: type π as pi.

$h =$ _____

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

13. Solve the formula $A = \frac{1}{2}bh$ for b.

$b =$ _____

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

14. Solve the formula $A = \frac{1}{2}h(a + b)$ for b.

$b =$ _____

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

15. Solve the formula $S = P(1 + rt)$ for P.

$P =$ _____

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

16. Solve the formula $A = P + Prt$ for r.

$r =$ _____

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

Contextless Linear Equations. Solve each of these equations.

17. Solve the equation for the given variable:

$$\frac{9m - 3}{4} = -2$$

If your answer is a fraction, write it in fraction from and reduce it completely. Do NOT convert to decimals.

$m =$ _____

18. Solve $1(x + 7) - 4 = 5(x - 2)$ for x algebraically. If your answer is a fraction, write it in reduced, fractional form. Do NOT convert the answer to a decimal.

$x =$_____

Identifying Linear Models. For each of the following equations whether in context or not, determine whether it is linear.

19. Is the following equation linear and consequently produce a graph that is a straight line?

$y = x^2 + 10x + 15$

1. Yes
2. No

20. Is the following equation linear and consequently produce a graph that is a straight line?

$8x + 14y = 11$

1. Yes
2. No

21. Is the following equation linear and consequently produce a graph that is a straight line?
$4x = 19 - 18y$

1. Yes
2. No

3.2 Representing Data

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)
- Identify data varying directly or indirectly (critical thinking)

We often represent numerical data using tables, diagrams, and graphs. These include various kinds of charts like bar graphs and pie charts, and graphs of functions. We do this to make certain traits of the data easier to notice. Here we will look at how some of these are produced and begin to learn to recognize differences due to rates. More details about rates will be covered in Section 3.3.

3.2.1 Reading Tables of Data

This section illustrates how some data can be stored in tables, how to read data from a table, and how to infer additional data when reasonable.

Tables are useful if we have a limited number of entries, and the data can be organized by two traits. If there are too many entries, the table may be too large to efficiently use. The two traits become headers for the rows and columns. If the table has more entries than we can see on a page or screen, it becomes less easy to use. Technology can make it easier to find the desired row and column such as a spreadsheet with the row and column headings frozen.

Table 3.2.1 Stall speed at 2550 lbs, most rearward center of gravity, speeds KIAS

	Angle of Bank			
Flap Setting	0°	30°	45°	60°
Up	48	52	57	68
Approach	43	46	51	61
Landing	40	43	48	57

We do not need to understand what bank, flap, and stall speed mean to read this table. Indeed a table can be presented precisely to help explain what terms mean. However, to satisfy your curiosity, stall speed refers to the speed at which a wing will produce insufficient lift to keep a plane flying. Falling beneath this speed typically results in the plane lowering its nose to regain speed. Angle of bank refers to how steeply the plane is tipped (left or right) in order to turn. KIAS stands for knots indicated air speed. Indicated airspeed is a speed pilots can see (think speedometer). Flaps are a structure extended for landing and sometimes take-off. Up means they are not in use. Approach and landing refer to varying degrees of extension.

Example 3.2.2 What is the stall speed in a 30° bank angle with flaps up?

We can determine this by looking for the column labeled "30°" and the row labeled "Up". In that cell is the number 52. Thus the stall speed at that bank angle with flaps up is 52 KIAS. ▲

Example 3.2.3 In what condition is the stall speed the highest?

If we read all three rows, the largest number we find is 68. That is in the Up row and 60° column. So the stall speed is highest in the steep, 60° turn with the flaps in the up position.

Note, there is no shortcut here for checking the entry in every row and column. ▲

Example 3.2.4 As the angle of bank increases (from 0° to 60°) what happens to the stall speed?

Solution. If we look in the Up row, the stall speed changes from 48 to 52 to 57 to 68. The first thing we notice is that the stall speed increases.

If we repeat this in the Approach row, we again see the speeds are increasing. The same is true in the Landing row.

Thus we can say that stall speed increases as the angle of bank increases.

In later sections (Section 3.3) we will learn to be more specific about patterns when possible. ▲

Checkpoint 3.2.5

	0	30	45	60
Up	48	52	57	68
Approach	43	46	51	61
Landing	40	43	48	57

Based on the table, what is the stall speed in a 30 degree turn with flaps in the Approach position? ___

Sometimes we want to know data that is between entries in a table. We can estimate these values if we know or can safely assume some property about the data. This is called **interpolation**. Below we provide examples of interpolation for linear data. Linear data is described in Section 3.1 and Section 3.3.

Example 3.2.6 Interpolation in a Table. What is the stall speed in a 15° bank angle with flaps in the Approach setting?

First, we note that there is no column for 15° bank angle. However we have 0° and 30°. 15° is half way between these two. For this chart it is reasonable to estimate our desired stall speed by calculating the number half way between those in the table.

The two stall speeds are 43 and 46. The number in between (the average) is $(43+46)/2 = 44.5$. Airspeed is reported only to units, so we must round to units. For stall speeds, it is safest to assume a higher stall speed, so we will round to 45 KIAS. ▲

When we want a value that is half way between two entries in a table, we can simply average them. However, if we want a value somewhere other than half way in between we must perform an additional calculation.

Example 3.2.7 What is the stall speed with 10° bank angle with flaps in the Up setting?

The nearest entries in the table are 0° and 30°. We want to measure how far 10 is from the left (0); we can calculate that as a percent of the distance from 0 to 30. We can use the percent to calculate the matching number between the table entries (48 and 52).

Percent is part/whole which in this case is

$$\frac{10}{30-0} = \frac{10}{30} = \frac{1}{3}.$$

We do not need to write this as a percent (it would be approximately 33.3%), because we are just using it to multiply in the next step.

We want the speed that is 1/3 of the way from 48 to 52. Again we treat this as a percent problem.

$$\frac{1}{3} \cdot (52-48) = \frac{4}{3}.$$

This result is how far above 48, so the speed is

$$48 + 4/3 \approx 49.3 \approx 50.$$

Again we round up for safety.

Because there are only 4 knots between the entries, it hardly seems worthwhile to do this work, especially because we round up for safety. There are times however, when this process is useful. ▲

Checkpoint 3.2.8

	0°	30°	45°	60°
Up	48	52	57	68
Approach	43	46	51	61
Landing	40	43	48	57

Based on the table, what is the stall speed in a 52.5 degree turn with flaps in the Landing position? ___

Flight safety requires rounding up stall speeds. As a result, round up to the nearest integer.

3.2.2 Reading Graphs

This section illustrates how some data can be represented in graphs and how to read data from a graph, including some comparisons of graphs.

Graphs that are curves (like lines) are read by finding a vertical heading that matches our question (think row) and reading the corresponding horizontal heading (think column). Note this could be reversed, that is, find a horizontal heading that matches and read the corresponding vertical one.

Example 3.2.9 Figure 3.2.10 presents the maximum engine out glide for an airplane. If that plane is 2400 ft above the ground, how many nautical miles (nm) can it glide forward?

First we note that 2400 ft matches the vertical axis. We want to find the line across the graph that represents 2400 ft. We notice that no line is labeled 2400. We do have 2000 and 4000 and there are lines between them. To figure out which of these lines we should use, we must figure out how many feet each minor line represents.

We count 10 minor lines between each major line. Because each major line represents 2000 ft, we know the minor lines represent $2000/10 = 200$ ft.

Because $2400 = 2000 + 2(200)$ we want the second minor line above 2000. We follow that to the blue line, then we follow the gray (minor) line down to the bottom. Our result is two minor lines before 4.

We must figure out how much each minor vertical line represents. Each vertical major line is 2 nm. Again there are 10 minor lines between each major line, so we know the minor lines represent $2/10 = 0.2$ nm.

The glide distance for 2400 ft is therefore $4 - 2(0.2) = 3.6$ nm. We subtracted here because it is before 4. ▲

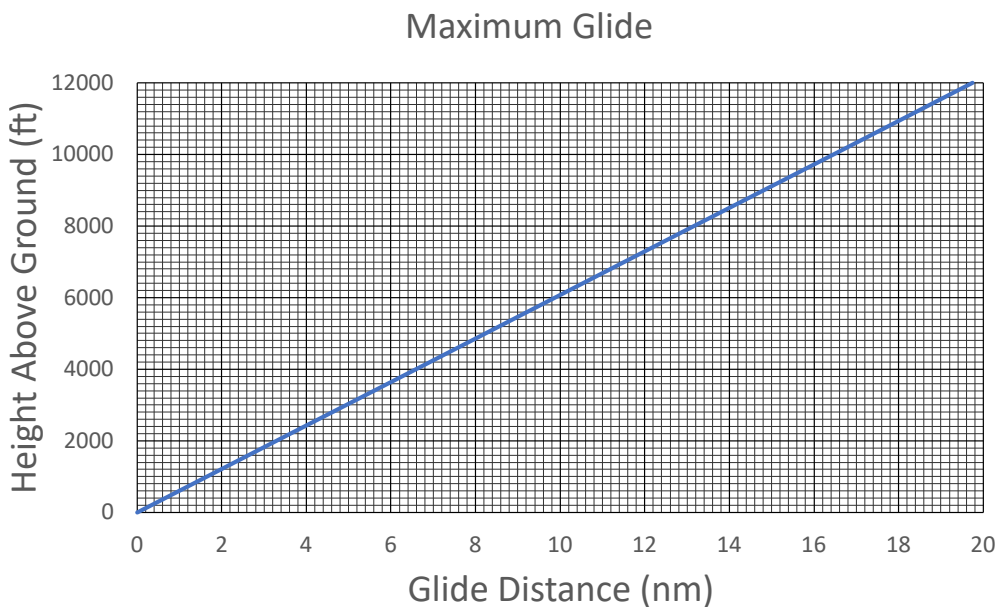
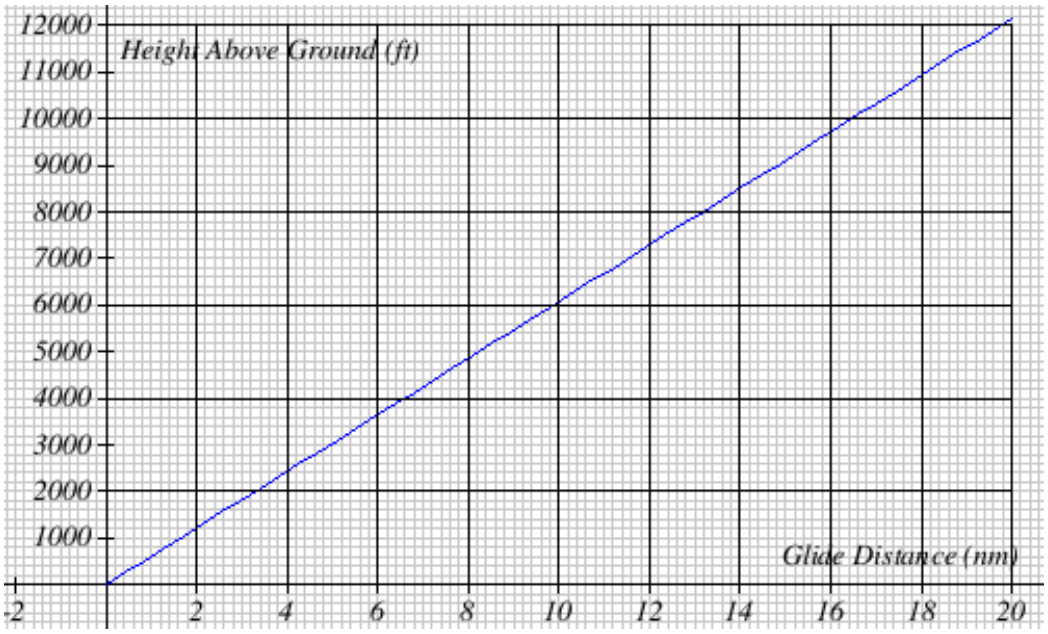


Figure 3.2.10 Graph Representing Maximum Engine Out Glide

Checkpoint 3.2.11 Based on the chart (graph) how far can the plane glide if it is 4600 ft above the ground?

____



You may round to the nearest 0.2 of a nautical mile.

When we look at any table, graph, or figure, we should ask ourselves why various choices were made in the construction of the table, graph, or figure. We may need to ask someone with related knowledge for explanations of those decisions.

Example 3.2.12 Consider Figure 3.2.10. The input we use is "Height Above Ground (ft)". Frequently we place the inputs on the x-axis. Why was the y-axis chosen for the inputs here?

Consider that the inputs are *heights*. This is a y-axis concept, so it matches our expectations. Reading the graph is not affected by this choice.

Why do the inputs begin at 0 and end at 12000?

They begin at 0, because we are talking about a plane gliding to the ground. A plane must be above the ground (above 0) to glide.

They end at 12000 in this case, because this aircraft cannot fly higher than that altitude. We do not need data for cases that cannot occur.

Why are the inputs labeled every 2000 and the inputs every 2?

This is purely space available. If we put in more labels, the text for them would overlap. ▲

Graphs can be from raw data which may appear random. We still read these graphs the same way.

For the following example we need to define a **percentile**. A percentile indicates a level such that, that percent of cases is less. For example if the 75th percentile is 88, means that 75% of the cases are less than 88. It is a way of indicating how much of a population is included at that level.

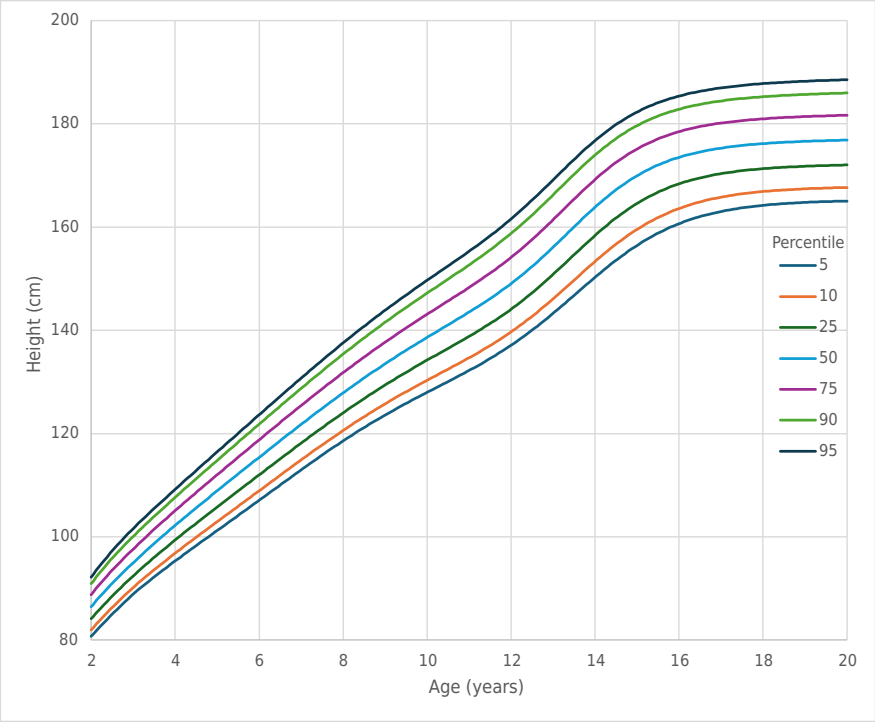
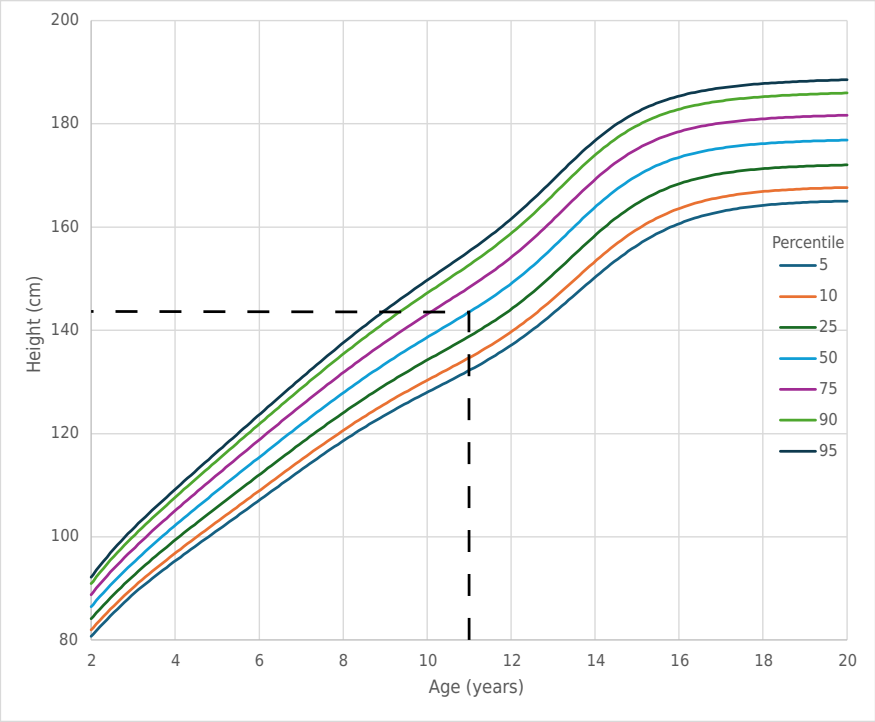


Figure 3.2.13 Expected Stature of Males from 2 to 20 Years

Example 3.2.14 The figure above contains the stature (height) percentiles for U.S. males. The data can be obtained at cdc.gov/growthcharts/. What height in centimeters is the 50th percentile for 11 year old males?

Age is on the x-axis, so we start at 11 which should be half way between 10 and 12. To read an approximate value from this figure, we can draw a vertical line from here up to the middle (blue) curve. This is shown on the image below. From that point we can draw a horizontal line across to the y-axis. This line is about one quarter of the way between 140 and 160, so we estimate (interpolate) this is 145 cm (see Example 3.2.7 for a reminder on how to estimate this).



How much greater is the spread (distance from 5th to 95th percentile) for 20 year olds than for 2 year olds? Present as a ratio and as a percent increase.

For 2 year olds it appears to range from 80 to 95 cm (about 3/4 of the way up). This is a spread of $95-80=15$ cm. For 20 year olds it appears to range from 165 to 190 cm. This is a spread of $190-165=35$ cm. Thus the spread increases over time. The ratio is $35/15 = 7/3 \approx 2.3$. For the percent increase we calculate $\frac{35}{15} \approx 2.33333333$ The percent increase is $2.33333333 - 1.00000000 = 1.33333333$ or 133% increase. Yes, the increase is greater than 100% in this case, that is, the spread more than doubled. ▲

In addition to reading specific values from these graphs, we should always ask ourselves why the diagram was designed as it was.

Example 3.2.15 When we look at the stature percentiles figure we should ask ourselves questions like the following.

Why is it for males only? The CDC provides separate data for females. On average females in the U.S. are shorter than males. Thus separate figures are needed. Note, to answer this we needed to see rest of the CDC data (cannot be read from the figure above).

Why does it begin at 2 years? The CDC provides separate data for 0-2 years. This is from the WHO rather than the CDC. An explanation is on the CDC website.

Why does the data end at 20 years? Notice that the percentile curves level off toward 20 years old. This simply represents that full, adult height has been reached. The curves should remain the same until age brings shrinking. ▲

Example 3.2.16 Figure 3.2.17 has the temperature and dewpoint read by a radiosonde (instruments on weather balloon) as it rose in the atmosphere. Note the vertical axis is the pressure reading. This is not the same as altitude, but it does correspond mostly to altitude. Dew point is the temperature at which water will condense, so it is also a temperature.

What are the temperature and dewpoint at the 700 millibar level?

We follow the 700 mb line over to the dewpoint (green, dashed) line then move straight down to the x-axis (temperature). This is a little less than halfway between -20° C and 0° (closer to 0°). We estimate -7° C. For temperature we continue across the 700 mb line to the temperature (red, solid) line then move straight down to the x-axis. We notice it is a little closer to 0° than the dewpoint. We estimate the temperature is 5° C. ▲

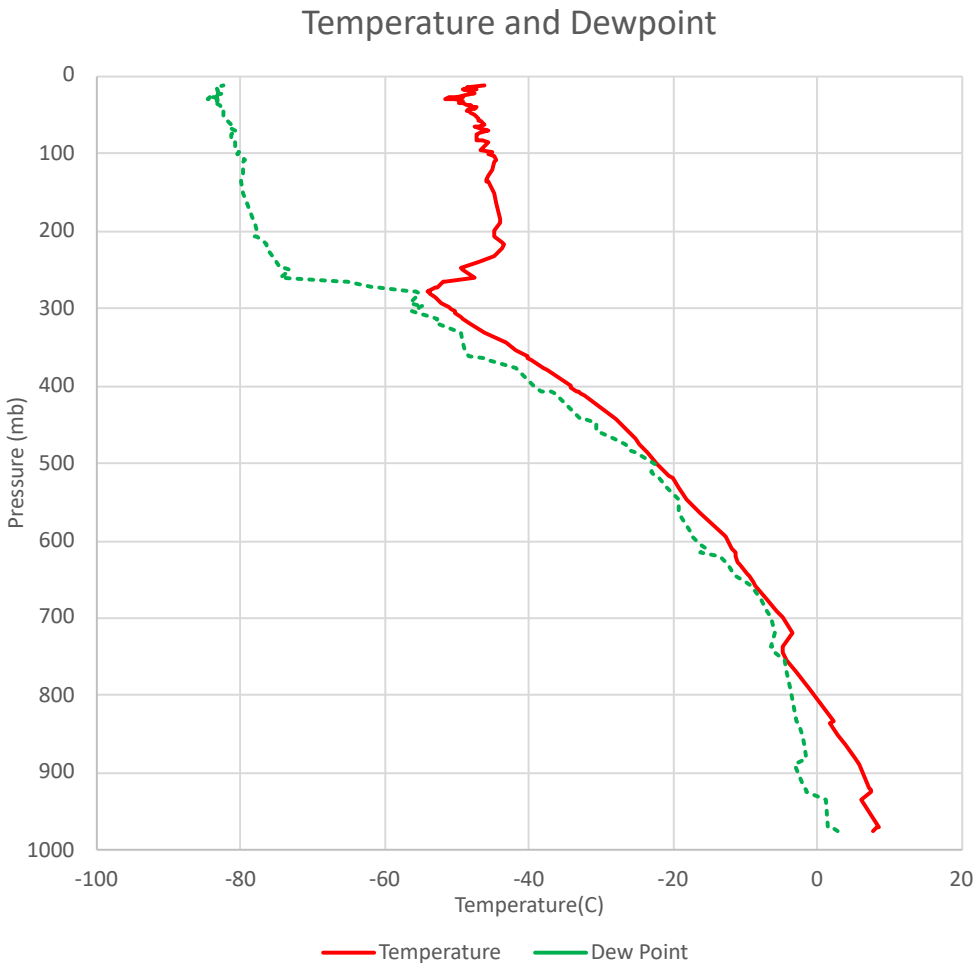


Figure 3.2.17 Graph of Temperature and Dewpoint

Note some charts like Figure 3.2.17 are not meant to convey specific numbers but rather to show trends.

Example 3.2.18 Notice that while the temperature (red, solid line) wiggles around, it trends down (to the left) as the pressure decreases (go up the graph). That is, it shows temperature generally decreasing as altitude increases. We expect this, because it is farther from the ground which heats the air. The increase above 300 millibars level is the result of other factors which you can research at `noaa.gov/jetstream/atmosphere/layers-of-atmosphere`.

Clouds form when the temperature reaches the dewpoint and the air is saturated (has enough moisture). We see in Figure 3.2.17 three places where temperature and dewpoint are the same. The lowest is between 800 and 700 millibars (we estimate 750 millibars). The second is between 700 and 600 millibars (we estimate 650 millibars). The third is at about 500 millibars. We would expect clouds to form at these altitudes. ▲

Example 3.2.19 Consider Figure 3.2.17. The input we use is "Pressure (millibars)". Why was the y-axis chosen for the inputs here?

The pressure readings correspond to altitudes (height) which we tend to think of as up. Putting this on the y-axis matches this expectation.

Why do the y-axis labels decrease as they go up?

Atmospheric pressure decreases with altitude, so low pressure means higher altitude. The pressure readings are arranged to be low altitude at the bottom and high altitude at the top.

Why do the inputs begin at 1000?

Because the pressure readings correspond to altitude the highest pressure should be on the ground. It turns out 1013 is a typical pressure at ground level, so much higher pressure readings are not expected.

Why are the output labels from -100 to 40?

These are based on commonly experienced temperatures. Temperatures lower than -100° C are not expected. Temperatures above 40° C do occur, but not in the location where this sounding was taken.

Use Example 3.1.8 to convert 50° C to Fahrenheit to see why this temperature is uncommon in most locations. ▲

The input for the glide ratio questions is altitude. Altitudes are **continuous** that is it makes sense to refer to an altitude of 2453.27 feet (fractional feet). Similarly the pressure levels are continuous, that is it makes sense to refer to 501.7 millibar level. However, there is data where a fraction does not make sense. This **discrete** data is often graphed differently. The next examples illustrate a way of presenting discrete data.

Example 3.2.20 Increasing Income. When Vasya was hired in 2017, she was paid an annual salary of $62,347.23. Her work has been good, so each year she has received raises of $5000.00.

To represent this data we first need to calculate her salary for each year. We do this by starting with her initial salary, then for each year adding the $5000 raise to the previous year's salary. This is an **iterative** process. Table 3.2.21 contains the results. A table is an effective means to represent discrete data like this.

We will represent her salary over time using the bar graph in Figure 3.2.22. Notice the horizontal axis is labeled with years and the vertical axis is labeled in dollars. There is one bar for each year, because her salary was changed only once each year. Bar graphs are a good option for discrete data.

Consider the bar graph (ignore the table). Can you tell that Vasya's salary is increasing? Can you tell how much? How might you change the graph to make it easier to answer these questions? ▲

Table 3.2.21 Vasya's Salary

2017	$62,347.23
2018	$67,347.23
2019	$72,347.23
2020	$77,347.23
2021	$82,347.23
2022	$87,347.23
2023	$92,347.23
2024	$97,347.23

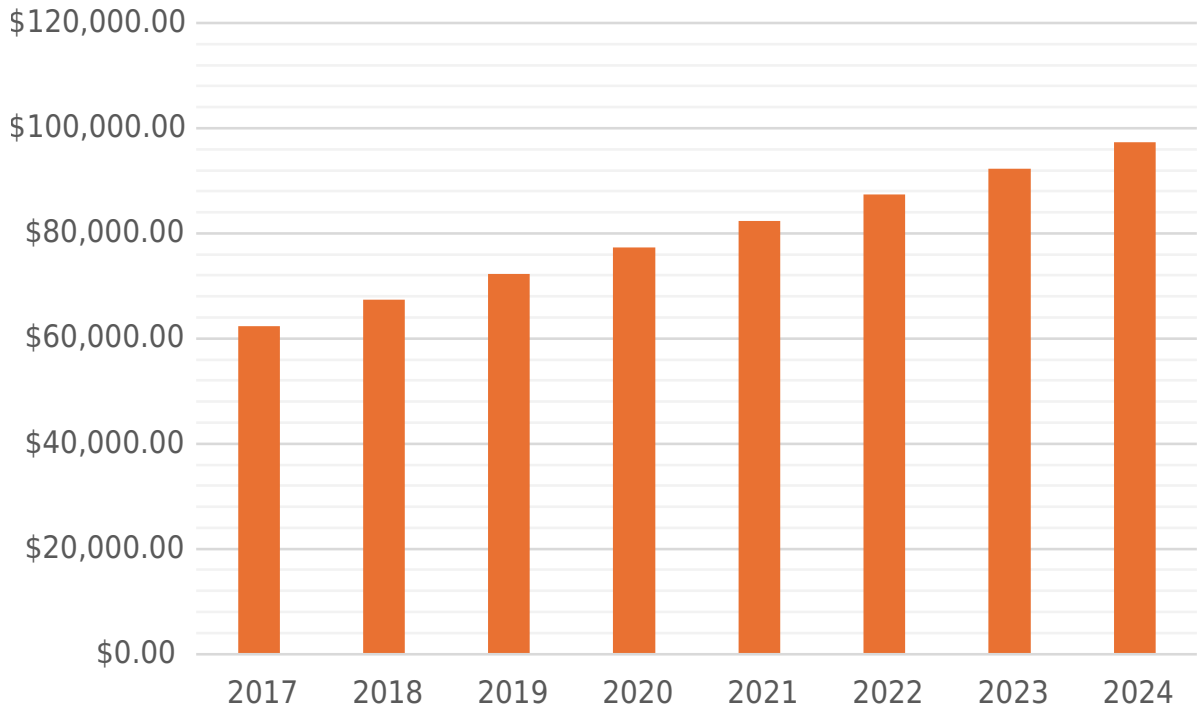


Figure 3.2.22 Vasya's Salary

Example 3.2.23 Vasya wishes to know how her raises are helping her keep up with increasing costs. Figure 3.2.24 shows her raises as a percent of her previous year's salary and the inflation rate (usinflationcalculator.com/inflation/current-inflation-rates/) If her raises are at least as large as inflation, then her spending power is not diminished.

(a) Using Table 3.2.21 confirm that the graph shows the correct percent increase for 2021. Recall her raise is \$5000 each year. The graph shows the percent increase for each year's salary relative to the previous year's salary.

(b) For these years is she keeping up with inflation?

Solution. Her raise is a larger percent each year except for two. In those two years it is close. She has been more than keeping up with inflation.

(c) What trend do you notice in the percent increase of salary? Why is this happening?

Solution. Her percent drops from about 8% to a little over 5%. This results from her raise being the same amount but her previous year's salary is bigger each year. In the percent (part/whole) the part is remaining fixed while the whole is increasing.

Unless there is a change this will lead to her raises eventually not keeping up with inflation.

▲

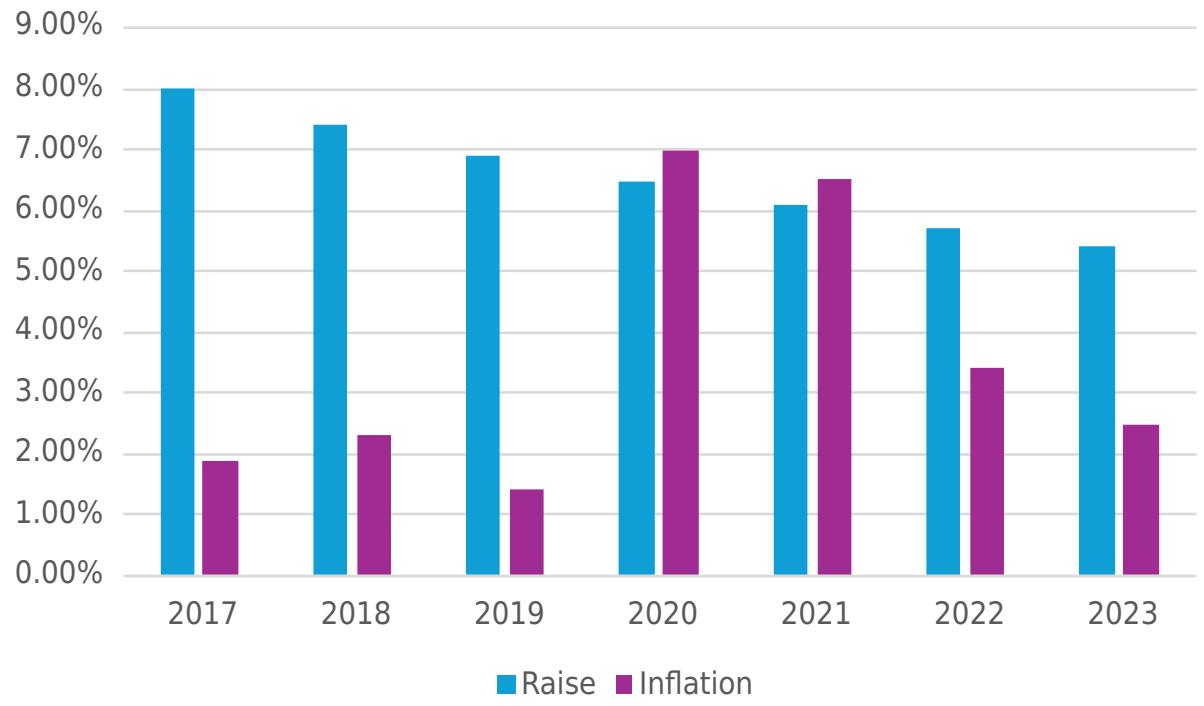


Figure 3.2.24 Vasya's Salary

Table 3.2.25 Inflation Rates by Year

2017	2.10%
2018	1.90%
2019	2.30%
2020	1.40%
2021	7.00%
2022	6.50%
2023	3.40%
2024	2.50%

Checkpoint 3.2.26 Consider Figure 3.2.27. It contains Guido's annual salary for each year listed. The second bar is the first year's salary increased each year to match inflation. That is it shows what Guido's salary would have been if his raises had exactly matched inflation.

(a) In which years did Guido receive a raise?

(b) In which years did Guido's salary appear to grow at least as much as inflation?

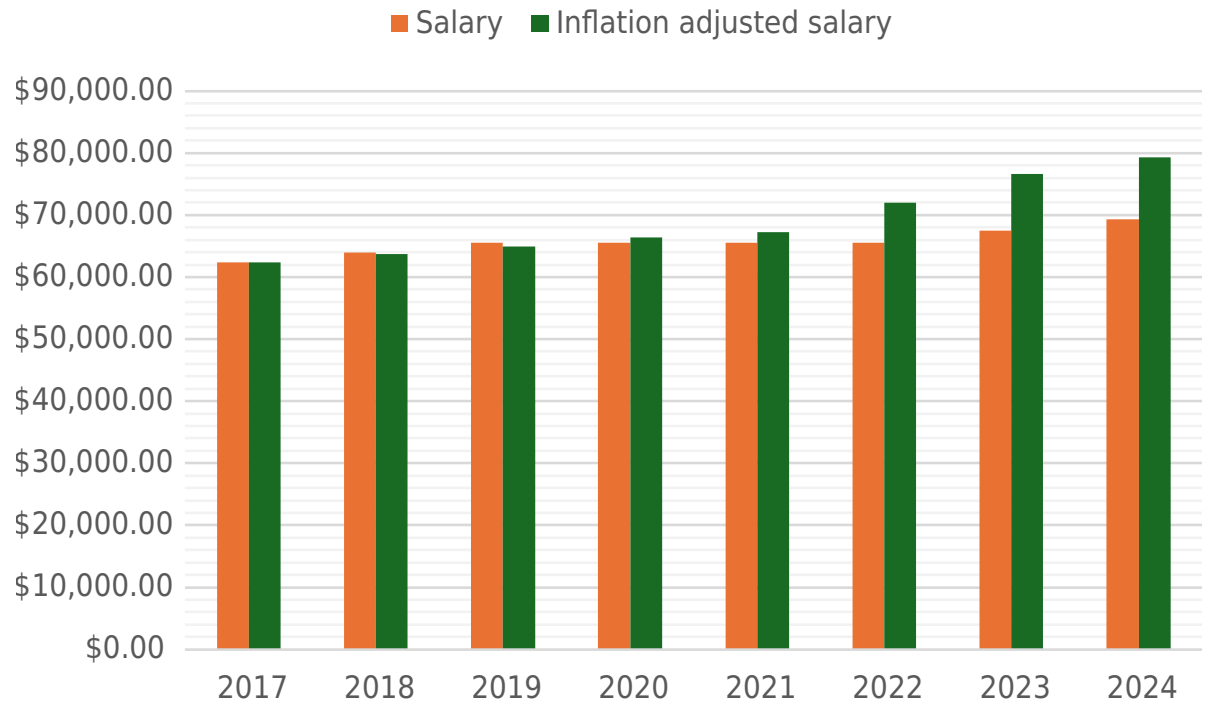


Figure 3.2.27 Salary vs Inflation

3.2.3 Using Graphs to Analyze Models

Above we practiced interpreting graphs provided for us. This section presents generating graphs to analyze and interpret models. While we will rely on technology to produce graphs, these examples begin with manual generation of graphs because that process helps us understand a model and it helps us understand what the graphs mean.

Example 3.2.28 Scale Model. Recall the scale model of a space shuttle in Example 2.4.3 which is labeled as 1:144. We calculated that a model part which is 1.72 inches represents an actual part that is $1.72 \cdot 144 = 247.68$ inches. In general $A = 144M$ where M is the size on the model and A is the size on the actual shuttle.

To represent this scale conversion as a graph we will generate a table like Table 3.2.21 then we will use that to plot the graph.

Model	Full Size
1.0	144
1.5	216
2.0	288
2.5	360
3.0	432

We sketch a graph by plotting the points first. Notice the five points based on the table above. Through the points we draw a curve: in this case it is a line. The graph is in Figure 3.2.29.

For the curious, software uses this same process to produce a graph. It usually plots a much larger number of points and then connects the dots with short line segments. ▲

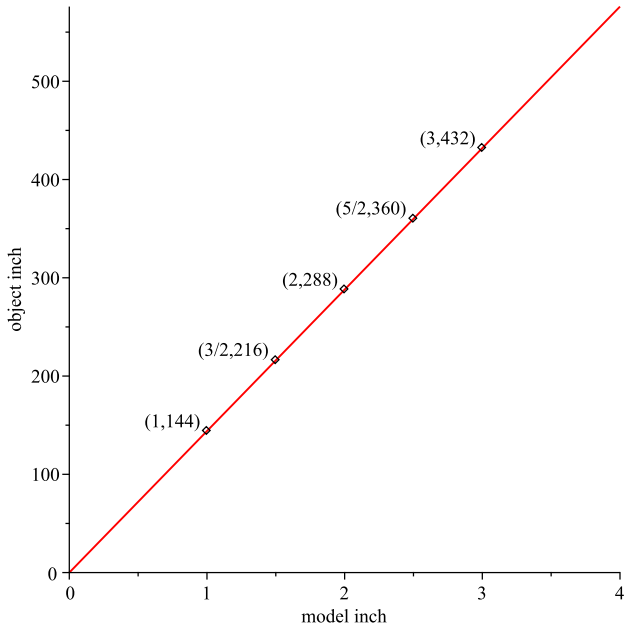


Figure 3.2.29 Graph of Scale

Example 3.2.30 Why does the graph start at 0? The inputs are lengths on the model; negative lengths do not make sense.

Why does it end at 4? If we wanted all sizes from zero to the largest dimension of the shuttle, we would need a bigger graph. However, because this is a line, we have a good idea what the rest of the graph looks like.

This graph is a line. We knew it would be because $A = 144M$ is in the form of a line (as shown in Subsection 3.1.1). ▲

The next example is a shape we have not yet encountered in this text.

Example 3.2.31 Ohm's Law. Recall Ohm's Law $V = IR$ from Model 1.3.1. We will explore the relationship between current (I) and resistance (R).

To begin the exploration and to enable graphing we will complete a table. First it will be convenient to solve Ohm's law for current (I).

$$\begin{aligned} V &= IR. \\ V \cdot \frac{1}{R} &= IR \cdot \frac{1}{R}. \\ \frac{V}{R} &= I. \end{aligned}$$

Because we are interested in the effect of resistance on current we will pick a fixed voltage: $V = 12$ V. Thus our equation is $I = \frac{12}{R}$.

Resistance	Current
1.0	12
4.0	3.0
8.0	1.5
12.0	1.0
16.0	0.75

We can plot these points and sketch a curve through them. This graph is in Figure 3.2.32.

The apparent relationship between current and resistance for a fixed voltage is that current decreases as resistance increase.

The graph starts with 1 Ohm. Why does it not start at 0? If resistance were 0, then the equation becomes $I = \frac{12}{0}$. Division by zero does not make arithmetic sense. 0 ohm resistance means no resistance and this is not physically possible (nothing is perfect). Thus the math model fits the physical reality. ▲

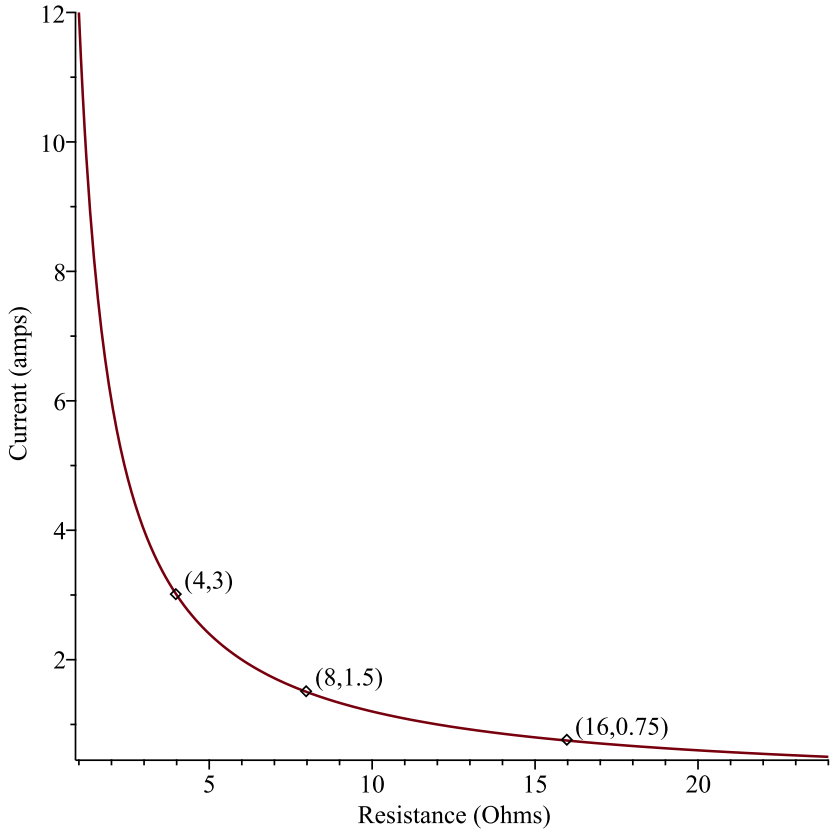


Figure 3.2.32 Graph of Ohms Law

Checkpoint 3.2.33 The ideal gas law expresses a relationship between pressure, volume, and temperature of a gas. It is given by

$$P \cdot V = k \cdot T$$

where P is the pressure, V is the volume, T is the temperature, and k is a constant dependent on the specific gas.

(a) Draw a graph for the equation $P = \frac{8.3145T}{2.0000}$. Note the units are Kelvin (Celsius + 273.15) for temperature and Jules/litre for pressure. These do not need to be labeled here.

(b) Draw a graph for the equation $P = \frac{8.3145 \cdot 293.15}{V}$.

3.2.4 Graphing Lines

We have seen what linear data looks like in data tables, discrete graphs (e.g., bar graph), and continuous graphs. This section presents how to graph lines when we have the equation, and presents analyzing linear models based on their graphs.

As described in Subsection 3.1.1 a linear equation (model) has two parts: the shift $+b$ and the ratio (slope, rate) m. First, we address the role of the ratio in the graph.

Definition 3.2.34 Slope. The rate of change of a line (graph) is called its **slope**. The numerator is the

change in y and the denominator is the change in x. Slope can be calculated as

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Rise and run are terms to help us remember the formula. ◆

In Example 2.4.3 we graphed a linear equation with the ratio 144 real inches to 1 model inch. In the linear model we have $m = \frac{144}{1}$ that is 144 is the change in y (real inches) and and 1 is the change in x (model inches).

Because the rate of change is the fixed, the slope can be calculated from any two points. We can calculate the slope from points in a table or points from a graph using

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

The next example illustrates calculating the slope from two points and that the slope is the same regardless of points selected.

Example 3.2.35 Calculate Slope. The graph in Figure 3.2.36 is linear. We will calculate the slope twice.

Solution 1. First, we will use the points $(4000, 25.92)$ and $(2000, 27.92)$.

$$\begin{aligned}\frac{y_2 - y_1}{x_2 - x_1} &= \frac{25.92 - 27.92 \text{ inHg}}{4000 - 2000 \text{ ft}} \\ &= \frac{-2.00 \text{ inHg}}{2000 \text{ ft}} \\ &= -\frac{1.00 \text{ inHg}}{1000 \text{ ft}}.\end{aligned}$$

This means the slope is a decrease of 1.00 inHg (inch of mercury) per increase of $100\bar{0}$ feet above mean sea level.

Solution 2. We will use the points $(8000, 21.92)$ and $(4000, 25.92)$.

$$\begin{aligned}\frac{y_2 - y_1}{x_2 - x_1} &= \frac{21.92 - 25.92 \text{ inHg}}{8000 - 4000 \text{ ft}} \\ &= \frac{-4.00 \text{ inHg}}{4000 \text{ ft}} \\ &= -\frac{1.00 \text{ inHg}}{1000 \text{ ft}}.\end{aligned}$$

As expected this is the same slope, because on a line the rate of change (slope) is constant. ▲

To write the equation for a line we need the shift, $+b$, as well as the slope. This can also be read from the graph. The slope determines how tilted the line is. After this that line can be moved up or down. b controls this shift. It is typically easiest to read this shift at $x = 0$, because in the linear form $y = b + mx$ we have $y = b + m \cdot 0 = b$. The shift then is the y coordinate at $x = 0$ which is why it is often called the y-intercept.

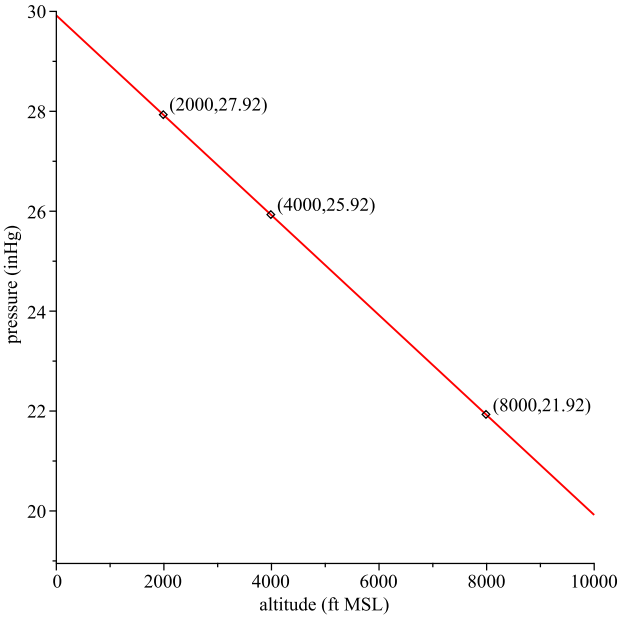


Figure 3.2.36 Calculating Slope

Example 3.2.37 Calculate Shift. The graph in Figure 3.2.36 is linear. We will calculate the shift.

Solution. The shift can be read when $x = 0$. That point is not labeled on the graph. However, we can calculate it using one of the points and the ratio.

We will use (2000 ft, 27.92 inHg) to calculate the pressure using a proportion. We want the point 2000 feet below this point, and pressure increases as we go down so we set up

$$\begin{aligned} \frac{1.00 \text{ inHg}}{1000 \text{ ft}} &= \frac{d \text{ inHg}}{2000 \text{ ft}}. \\ \frac{1.00 \text{ inHg}}{1000 \text{ ft}} \cdot 2000 \text{ ft} &= \frac{d \text{ inHg}}{2000 \text{ ft}} \cdot 2000 \text{ ft}. \\ 2.00 \text{ inHg} &= d. \end{aligned}$$

Thus the pressure should increase by 2.00 inHg giving us $P = 27.92 \text{ inHg} + 2.00 \text{ inHg} = 29.92 \text{ inHg}$. Thus the point is (0.00 ft, 29.92 inHg) and the shift is $b = 29.92$ inHg.

Combining this shift with the slope from the example above the model is

$$P = 29.92 \text{ inHg} - \frac{1 \text{ inHg}}{1000 \text{ ft}} A.$$

- P is the pressure at altitude A.
- 29.92 is the initial pressure.
- The rate is $\frac{1 \text{ inHg}}{1000 \text{ ft}}$.
- A is the altitude above ground level.

If we replaced 29.92 with a parameter P_G we could generalize the model to $P_A = P_G - \frac{1}{1000} A$. ▲

Checkpoint 3.2.38 Suppose that as dry air rose its temperature dropped in a linear fashion. The temperature was measured at 900 ft MSL as 10° C and at 3900 ft MSL as 4.3° C.

What is the rate of change of temperature with respect to altitude? _____

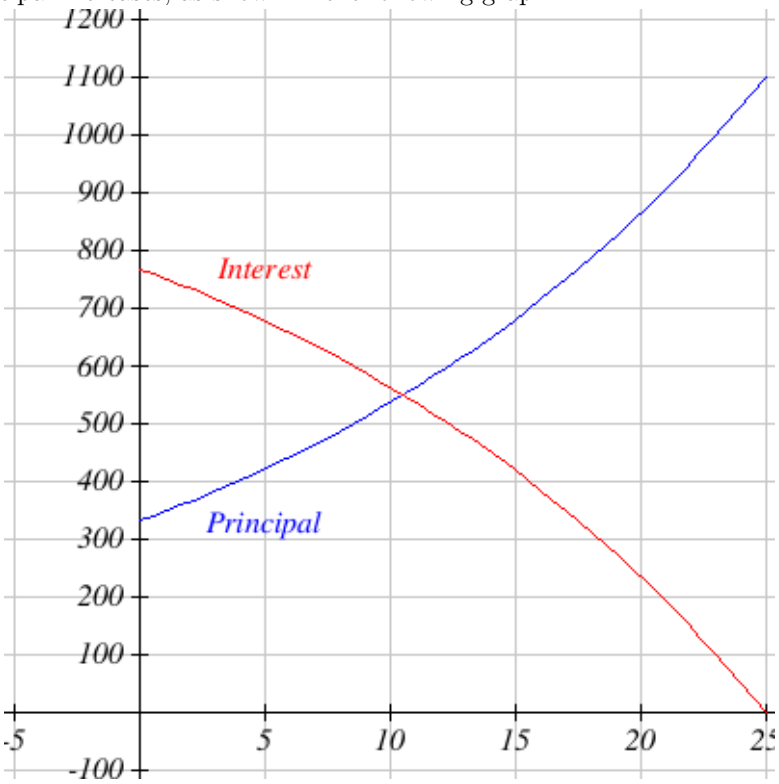
Your response may be as a fraction such as 2/1000 or decimal such as 0.002

What does this data imply about the temperature at 0 ft MSL? _____

3.2.5 Exercises

Interpret Charts. Answer the questions based on the charts provided.

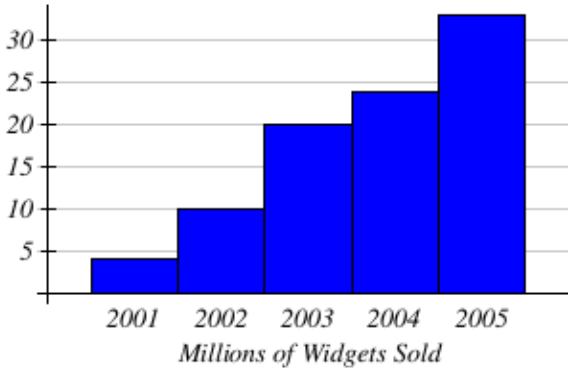
1. Becky takes out a 25-year mortagage for which her monthly payment is \$1100. During the early years of the mortgage, most of each payment is for interest and the rather small remainder for principal. As time goes on, the portion of each payment that goes for interest decreases while the portion for principal increases, as shown in the following graph:



a) Approximately how much of the \$1100 monthly payment goes for interest in year 10?
\$_____

b) In what year will the monthly payment be equally divided between interest and principal?
Year _____

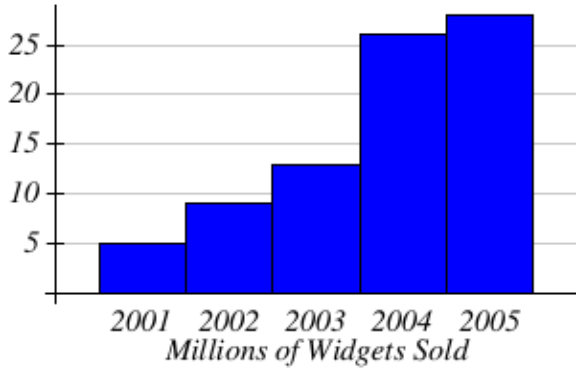
2. Our company's new widget has been growing in sales. The histogram below shows sales in millions for the years shown.



Approximately how many millions of widgets were sold in year 2002?

_____

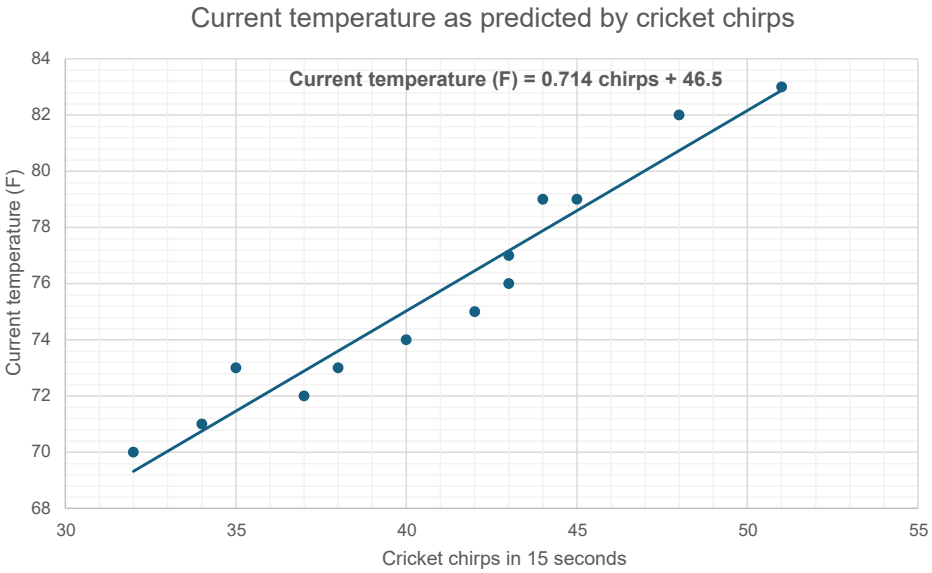
3. Our company's new widget has been growing in sales. The histogram below shows sales in millions for the years shown.



In which year did sales first exceed 7 million widgets?

_____

4. Part 1 of 4

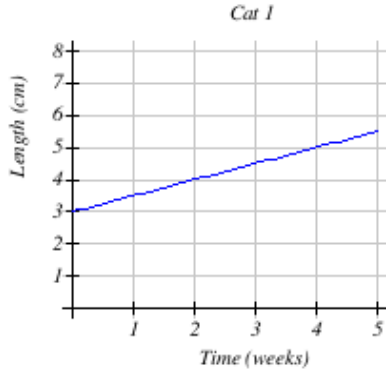


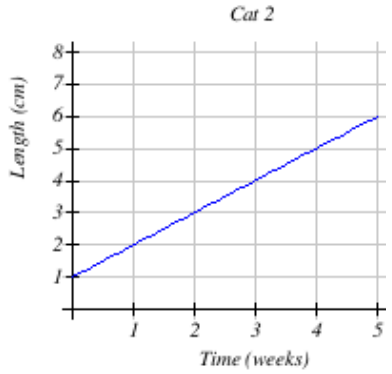
What questions and thoughts do you have based on viewing this chart?
Part 2 of 4
Which causation do you think is more likely?

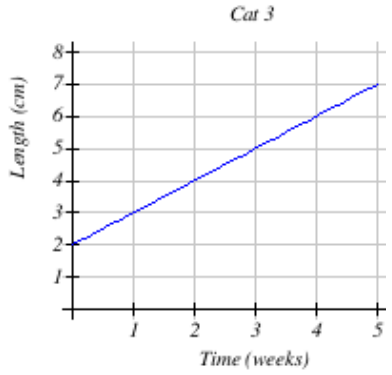
1. temperature changes the number of chirps
2. chirps change the temperature

Why?
Part 3 of 4
If you hear 41 chirps in 15 seconds, what temperature does this predict? ___
Part 4 of 4
How much to the left do you think is reasonable to extrapolate? Why?
How much to the right do you think is reasonable to extrapolate? Why?

5. An animal shelter once tracked the length of the whiskers of three cats every week starting from the time they received the cats. The three graphs are shown below.







At what rate are the whiskers of Cat 2 growing? ____

1. cm per week
2. weeks per cm
3. weeks
4. cm

Which cat's whiskers are growing more slowly than the other two?

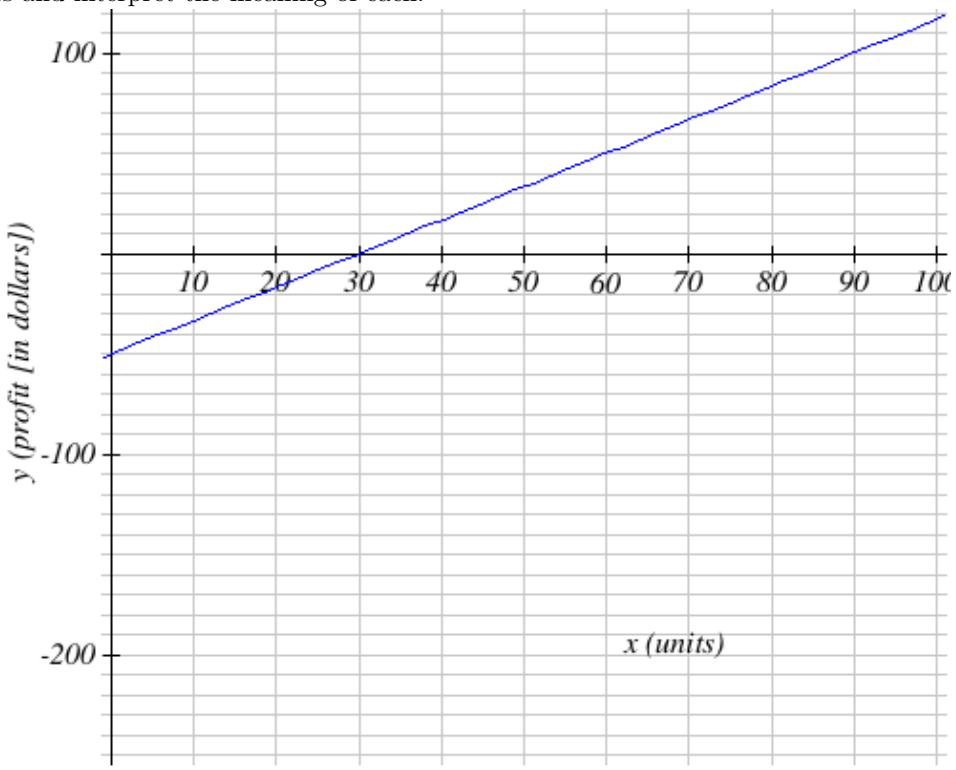
1. Cat 1
2. Cat 2

3. Cat 3

How long were the whiskers of Cat 3 when it was received by the animal shelter? ____

1. cm per week
2. weeks per cm
3. weeks
4. cm

6. In the graph below the input x is the number of units produced by a machine in a factory. The output y is the profit made by the sale of these units when they are produced. Determine the intercepts and interpret the meaning of each.



A. Give the coordinates of the x-intercept: _____
B. Interpret the meaning of the x-intercept:

1. The profit is \$30 when -50 display units are produced and sold.
2. The profit is zero when 30 units are produced and sold.
3. The profit is \$-50 when 30 display units are produced and sold
4. There are zero units produced and sold when the profit is \$30.
5. The profit is zero when -50 units are produced and sold.
6. There are zero units produced and sold when the profit is \$-50

C. Give the coordinates of the y-intercept: _____
D. Interpret the meaning of the y-intercept:

1. The profit is zero when -50 units are produced and sold.

2. The profit is zero when 30 units are produced and sold.
3. The profit is \$-50 when zero units are produced.
4. There are zero units produced and sold when the profit is \$30.
5. The profit is \$30 when -50 display units are produced and sold.
6. The profit is \$-50 when 30 display units are produced and sold

E. Determine the slope of the line: _____ [Make sure your slope is written as a reduced fraction.]

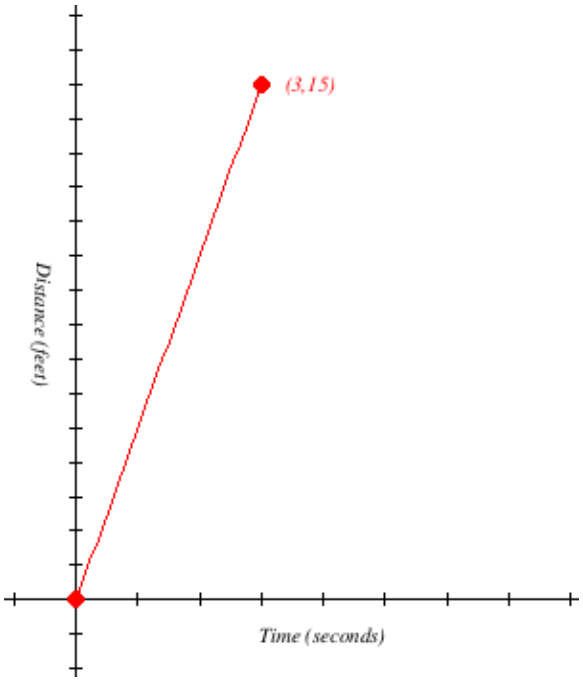
F. Interpret the meaning of the slope. For each 12-unit increase in units sold there is a(n)

1. decrease
2. increase

_____ *G.* Write the equation of the line in $y = mx + b$ form: _____

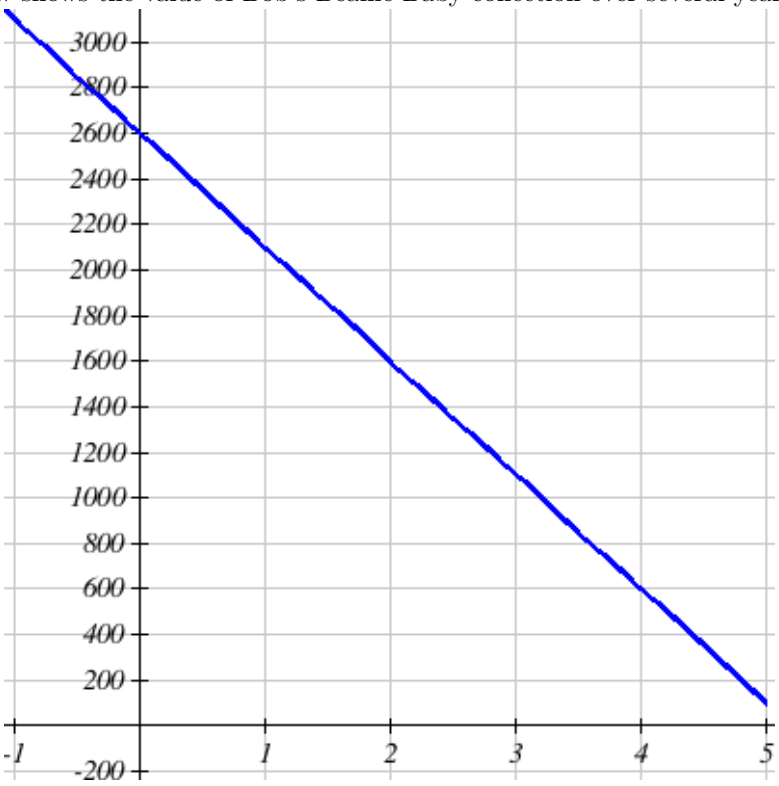
H. Use your equation to determine what quantity of units will yield a profit of \$190: _____

7. The graph below shows the distance a cat is from his bowl of milk over time. Which sentence is a good match for the graph?



1. The cat ran away from the milk at a rate of 3 feet per second.
2. The cat ran away from the milk at a rate of 5 feet per second.
3. The cat was 15 feet away from the milk and ran toward it reaching it after 3 seconds.
4. The cat was 3 feet away from the milk and ran toward it reaching it after 15 seconds.
5. The cat was 15 feet away from the milk and ran away from it at a rate of 3 feet per second.

8. The graph below shows the value of Bob's Beanie Baby collection over several years.

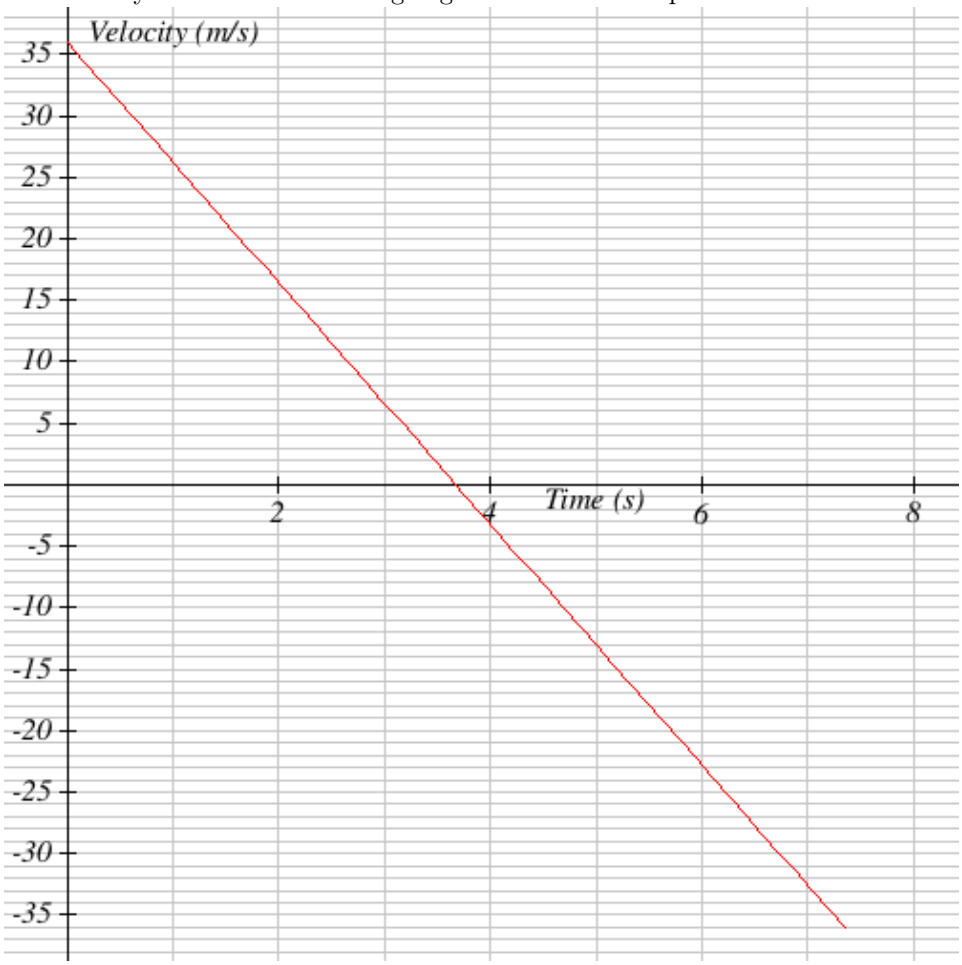


The collection's value decreased at a rate of _____...

1. dollars per Beanie Baby
2. Beanie Babies per dollar
3. dollars per year
4. years per dollar

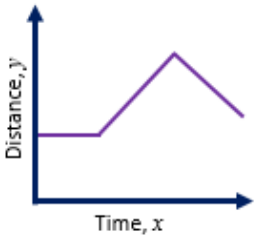
9. The graph shows the velocity of a ball that is thrown upwards and remains in the air for 7.35 seconds.

Negative velocity indicates the ball is going down instead of up.



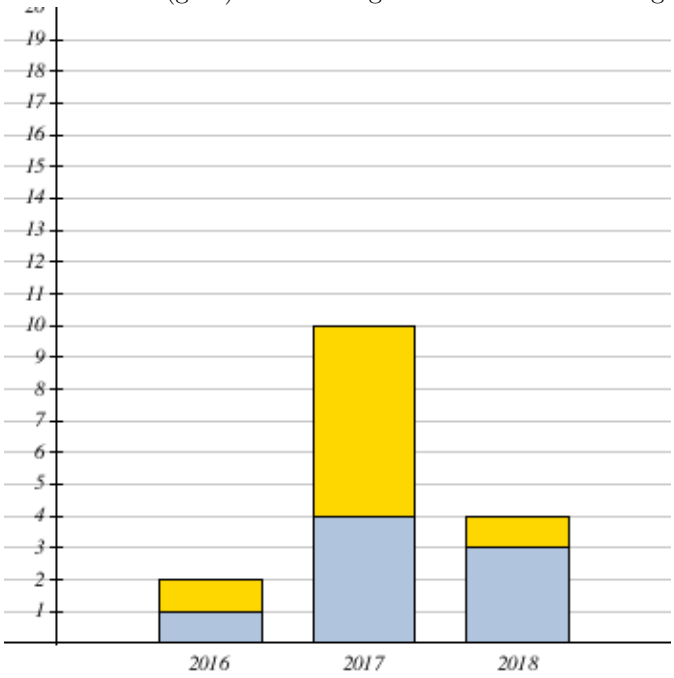
What is the final velocity of the ball?
$v_f =$ _____ unit_____
At approximately what time does the ball change direction?
$t_f =$ _____ unit_____

10. The graph shows the distance traveled by a vehicle over time. Describe the change in distance.



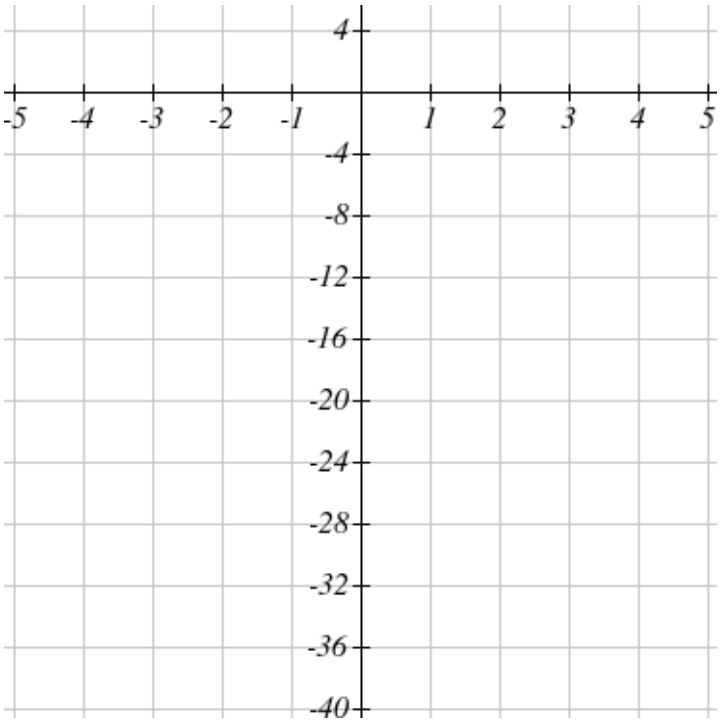
1. First the distance *increases* at a constant rate. Then the distance *stays the same* for a while before *increasing* at a constant rate.
2. First the distance *increases* at a constant rate. Then the distance *decreases* at a constant rate before staying the *same* for a while.
3. First the distance *decreases* at a constant rate. Then the distance *stays the same* for a while before *increasing* at a constant rate.
4. The distance *stays the same* for a while. Then the distance *increases* at a constant rate before *decreasing* at a constant rate.

11. The park service records the number of reported rattle snake sightings on Bald Mountain Trail (light blue) and Camel Back Trail (gold). The data gives rise to the following stacked bar graph.



(a) How many rattle snake sightings were there on Bald Mountain Trail in 2016? _____
(b) How many rattle snake sightings were there on Camel Back Trail in 2016? _____
(c) How many total rattle snake sightings were there in 2017? _____
(d) What percentage of rattlesnake sightings in 2018 were from Bald Mountain trail? _____%
Report your answer to at least two decimal places.

12. In Biology class, students store their specimen's for labs at -30 ˚C. To thaw, a specimen is brought to a refrigeration unit. After 1 hour, the specimen's temperature in the refrigeration unit is -23 ˚C. Sketch a graph to model the situation.



$m =$_____

1. It takes 30 minutes to warm up.
2. The specimen starts at 30 degrees below zero.
3. The slope is 30.
4. The temperature goes up 30 degrees per hour.

_____

1. The temperature goes up 7 degrees per hour.
2. The y-intercept is 7.
3. It takes 7 minutes to warm up.
4. The specimen starts at 7 degrees below zero.

13. The cost to take a taxi from the airport is a function of the distance driven. A 5 mile taxi ride from the airport costs \$10. The cost is \$17 for a 10 mile ride.

y is the cost and x is number of miles.

a. Write an equation to model the situation. $y =$ _____

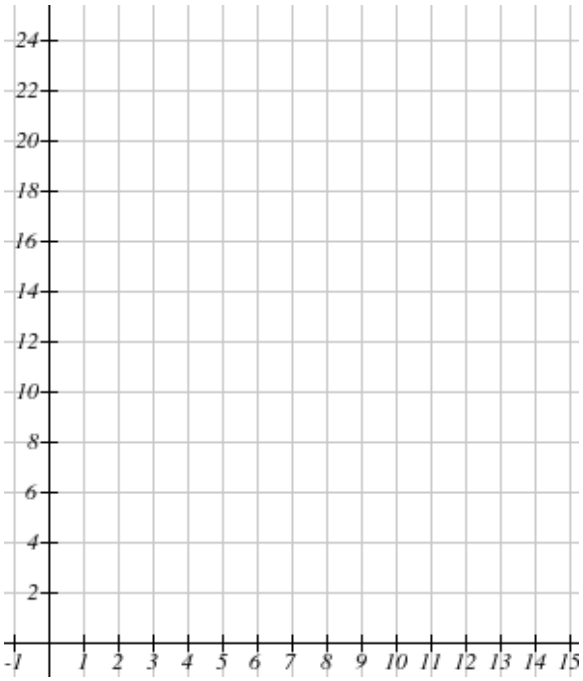
b. What is the y-intercept? $b =$ ____ What does it represent?

1. The cost for getting into the cab is \$3.
2. You start at \$10
3. The meter drop is \$1.40.
4. It takes 5 miles to get home.

c. What is the slope? $m =$ ____ What does it represent?

1. The meter drop is \$3.
2. You start at \$3
3. It takes 5 miles to get home.
4. It costs \$1.40 per mile.

d. Graph the equation.



14. In 1993, the moose population in a park was measured to be 3810. By 1999, the population was measured again to be 5370. If the population continues to change linearly:

Find a formula for the moose population, P, in terms of t, the years since 1990.

$P(t) =$ _____

What does your model predict the moose population to be in 2004?

_____

15. Select all of the following tables which could represent a linear function.

1.

x	$g(x)$
0	5
5	-10
10	-25
15	-40

2.

x	$h(x)$
0	5
5	30
10	105
15	230

3.

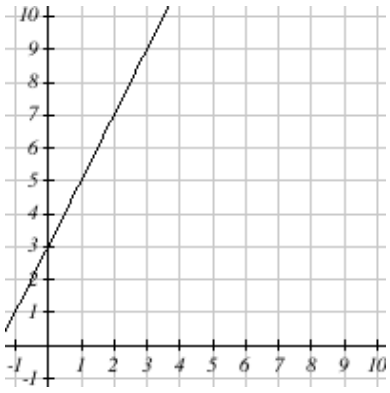
x	$f(x)$
0	-4
5	21
10	46
15	71

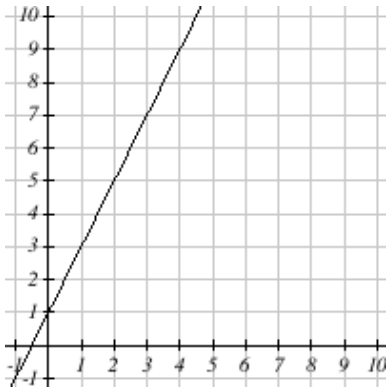
4.

x	$k(x)$
5	11
10	26
20	56
25	71

16. Below, one description, one graph, and one equation are equivalent. Choose the proper set.

1. You start with 1 Xbox games and each month you buy 2 new games.
2. You start with 3 Xbox games and each month you buy 3 new games.



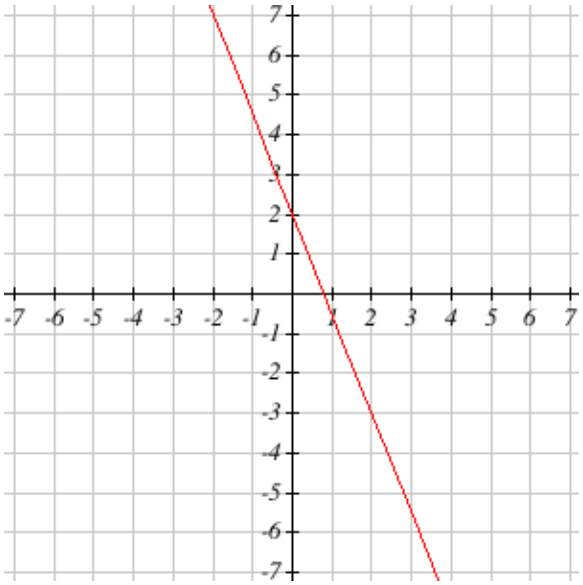


1. $y = 3x + 3$
2. $y = 2x + 1$

Rates. Answer these questions about rates/slopes of lines.

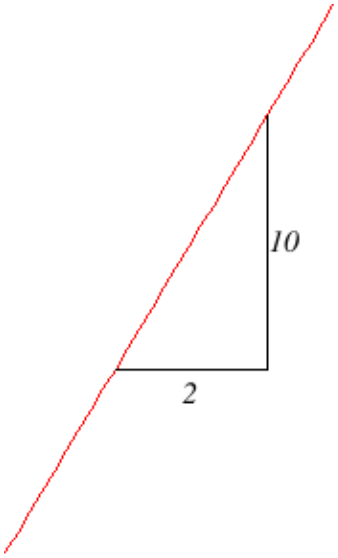
17. Find the slope of the line that goes through the points (7,-7) and (13,1).
Slope, $m =$ _____
Enter your answer as an integer or a reduced fraction in the form A/B

18.



Find the slope of the line.
Slope = m = _____
Enter your answer as an integer or as a reduced fraction in the form A/B.

19.



State the run, rise, and slope of the line above.
run = _
rise = _
m = _

20. Superman needs to save Lois from the clutches of Lex Luthor. It takes Superman 10 seconds to get to Lois who is 530 feet away. What is Superman's rate?

_____

1. s/ft or seconds per foot
2. ft/s or feet per second

21. You are on an oceanographic research expedition that began in San Juan, Puerto Rico on September 14.

The ship left port at 0630 hr on 14 September and covered a distance of 1835 km to the first drill location (Site 1) where you are going retrieve a drill core of seafloor sediments. The ship arrived at the first drill site at 1800 hr on 16 September.

Calculate the rate of travel (i.e., speed) of the ship during its transit to the first drill site. *Round your answer to the nearest tenth.*

_____ km/hr

22. A city's population in the year x =1987 was y =3,221,700. In 1972 the population was 3,223,200.

Compute the slope of the population growth (or decline) and choose the most accurate statement from the following:

1. The population is increasing at a rate of 100 people per year.
2. The population is increasing at a rate of 250 people per year.
3. The population is increasing at a rate of 50 people per year.
4. The population is decreasing at a rate of 250 people per year.
5. The population is decreasing at a rate of 100 people per year.
6. The population is decreasing at a rate of 50 people per year.

23. Determine Linearity and Slope from a Table

For each of the following functions, determine if the function is linear.

If it is linear, give the slope. If it is not linear, enter "DNE" for the slope.

x	6	11	15	17	18
$f(x)$	-37	-74.5	-104.5	-118	-127

Behavior:

1. Linear
2. Not Linear

Slope: _____

x	4	6	7	9	13
$f(x)$	34	47.5	54.25	67.75	94.75

Behavior:

1. Linear
2. Not Linear

Slope: _____

x	10	17	18	19	20
$f(x)$	-76	-125	-132	-139	-146

Behavior:

1. Linear
2. Not Linear

Slope: _____

x	6	8	14	19	20
$f(x)$	10	16	30	49	52

Behavior:

1. Linear
2. Not Linear

Slope: _____

24. Tyrone is babysitting to earn money to visit his aunt. He earns $25 for each hour of babysitting. Which graph(s) represent his earnings from babysitting? Mark all that apply.

1.



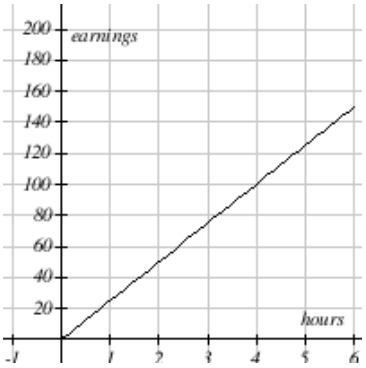
2.



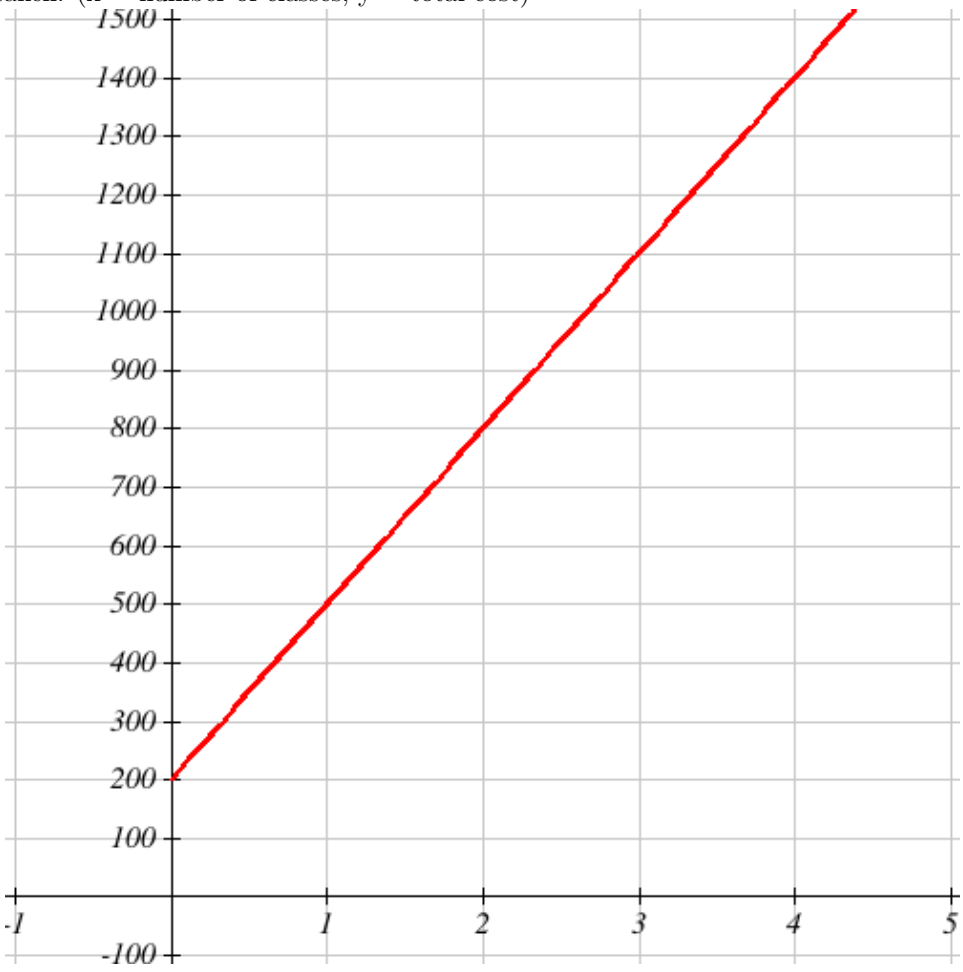
3.



4.



25. The graph below shows the total cost of attending a particular college based on the number of classes taken. (x = number of classes, y = total cost)



The slope of this line tells us that the cost of attending this college increases by _____

1. classes
2. dollars per class
3. dollars
4. dollars per year

$ _____

Write Linear Equations. Write an equation that matches the data provided.

26. Find the equation (in terms of x) of the line through the points (-5,-1) and (3,2)

$y =$ _____

27. Write the equation in slope-intercept form of the line that has slope 1 and y-intercept $(0, -9)$.

$y =$ _____

28. Given the table of Celsius and corresponding Fahrenheit temperatures, find the linear relationship where Celsius, c, is the input and Fahrenheit, f, is the output.

Celsius	Fahrenheit
-10	14
0	32
10	50
20	68
30	86
40	104
50	122
60	140

_____

29. Find the constant rate of change described from the table.

hours	dollars
6	113
11	63
16	13
21	-37

Rate/Slope: _____

1. dollars per hour
2. dollars
3. hours per dollar
4. hours

30. Suppose you start with a full tank of gas (19 gallons) in your truck. After driving 5 hours, you now have 6 gallons left.

If x is the number of hours you have been driving, then y is the number of gallons left in the tank.

At what rate is the gas left in the tank changing? State your answer as a reduced fraction.

_ gallons per hour

Find an equation of a line in the form y = mx + b that describes the amount of gas in your tank.

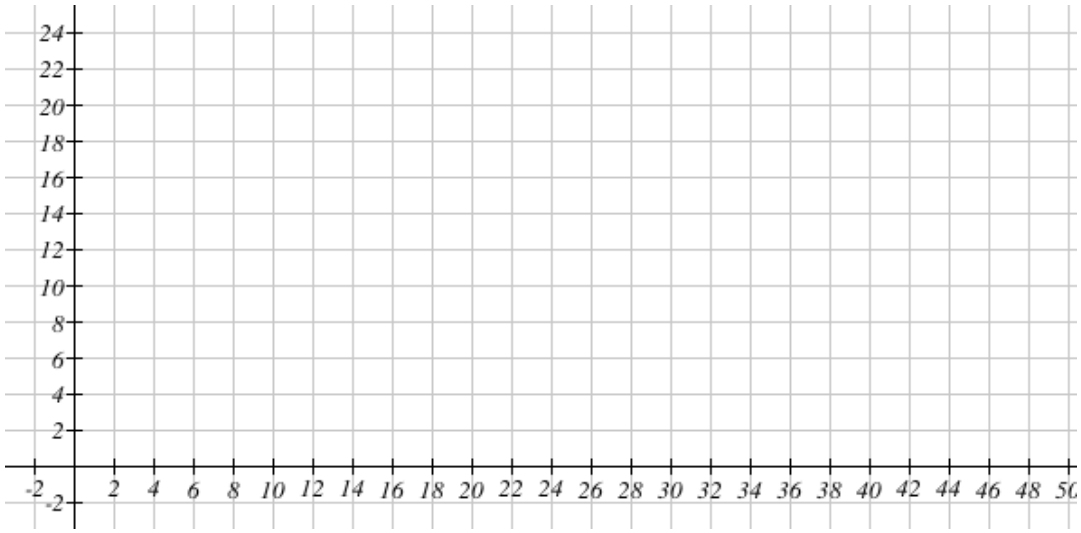
$y =$ _____

31. Prince John has 12 bags of gold. While he is sleeping, Robin Hood is stealing one bag every two minutes.

1. Write an equation in slope-intercept form to model the situation, where B is the number gold bags and t is the time in minutes.

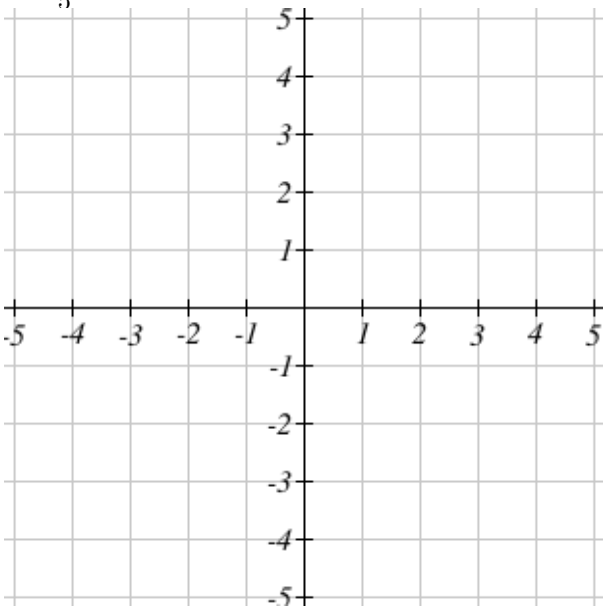
   $B =$ _____

2. Graph your equation below.

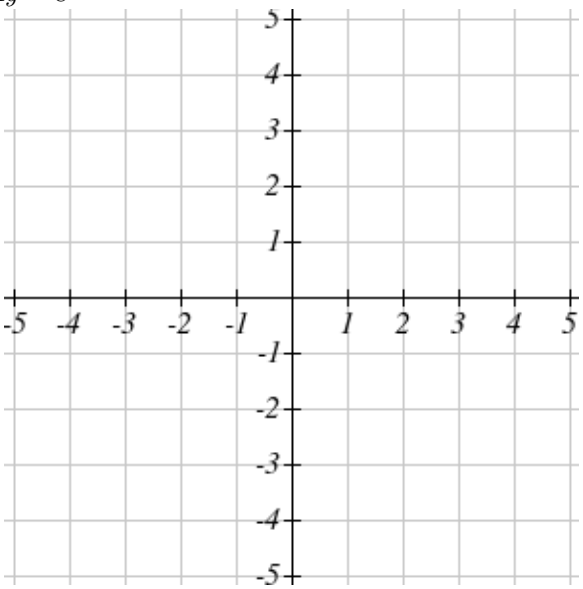


Graphing. Graph each line.

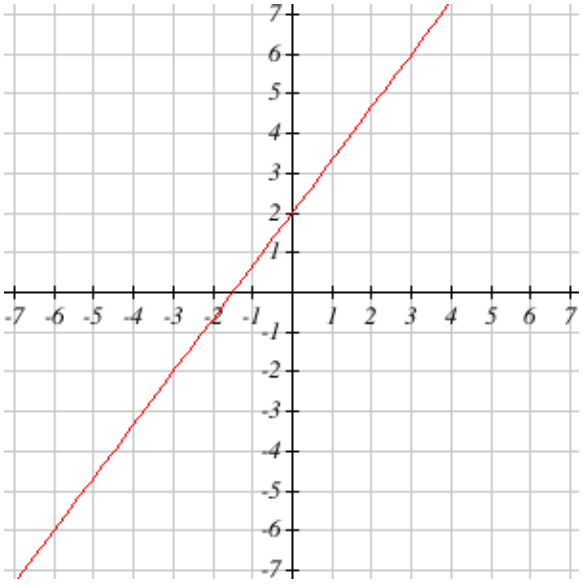
32. Graph the equation $f(x) = \frac{4}{5}x + 4$.



33. Sketch a graph of $8x + 4y = 8$



34.



What is the slope of the graph? Leave your answer in simplest form.
Slope = ___
Identify the y-intercept.
y-intercept = (0, __)
Write an equation in slope-intercept form for the graph above.
$y =$ _____

3.3 Identifying Rates

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)

So far we have looked at linear models. We will add quadratic, exponential, and some variations in later sections. One of the ways we distinguish between models is by the rate at which they grow. Often the rate at which something is happening is more important than how much there currently is. This section presents two methods for identifying rates from tables of data. How to identify each type by graph is presented in the appropriate chapter and section.

3.3.1 Differences

One way to measure rates is to look at the differences between data points. Calculating these differences is illustrated in the table below.

Table 3.3.1 Tien's Salary

Year	Salary	Difference
2017	$52,429.33	
2018	$55,050.80	$55,050.80-$52,429.33=$2,621.47
2019	$57,803.34	$57,803.34-$55,050.80=$2,752.54
2020	$60,693.50	$60,693.50-$57,803.34=$2,890.17
2021	$63,728.18	$63,728.18-$60,693.50=$3,034.68
2022	$66,914.59	$66,914.59-$63,728.18=$3,186.41
2023	$70,260.32	$70,260.32-$66,914.59=$3,345.73
2024	$73,773.33	$73,773.33-$70,260.32=$3,513.02

In order to see how these differences can help us distinguish between linear and other models, consider Vasya's salary in Example 3.2.20. We know that the difference between each year's salary is $5000.00, because we are told that was the raise each year. This is linear model. In contrast Tien's raises are different each year (they grow year to year). This means his salary does not grow linearly.

Example 3.3.2 Differences for Atmospheric Pressure Model. Consider the model in Example 3.2.35. Table 3.3.3 calculates the differences every 2000 ft. Notice that the differences are all the same namely -2.00. This model is linear.

Table 3.3.4 calculates the differences every 1000 ft. In this table the differences are all -1.00. This still indicates that the model is linear. It does not matter what interval we choose. If the differences over evenly spaced intervals are the same, then the model is linear.

The rates we obtained in both tables do match. Consider $\frac{-2.00}{2000} = \frac{-1.00}{1000}$. ▲

Table 3.3.3 Atmospheric Pressure Differences (2000 ft intervals)

Altitude (ft MSL)	Expected Pressure (inHg)	Difference
0	29.92	
2000	27.92	27.92-29.92=-2.00
4000	25.92	25.92-27.92=-2.00
6000	23.92	23.92-25.92=-2.00
8000	21.92	21.92-23.92=-2.00

Table 3.3.4 Atmospheric Pressure Differences (1000 ft intervals)

Altitude (ft MSL)	Expected Pressure (inHg)	Difference
0	29.92	
1000	28.92	28.92-29.92=-1.00
2000	27.92	27.92-28.92=-1.00
3000	26.92	26.92-27.92=-1.00
4000	25.92	25.92-26.92=-1.00

Definition 3.3.5 Linear Relation. A relation is **linear** if and only if the rate of change is constant. ◆

This states that a linear model grows by the same amount from one step to the next (rate of change or difference). This equal growth results from the ratio m in the form $y = mx + b$. Consider the specific case $y = 3x + 2$. In the table below notice that the differences are all the same and that the difference is 3. 3 is the ratio from the equation. This is always the case. The slope is how fast the line grows.

Table 3.3.6 Differences for Lines

x	y	Difference
0	2	
1	5	5-2=3
2	8	8-5=3
3	11	11-8=3
4	14	14-11=3

Notice we used this constant addition property when working with ratio problems like Example 2.3.4. Relations defined by fixed ratios like these are linear.

For linear data consecutive differences are always the same. The next examples (Example 3.3.7 to Example 3.3.10) illustrate known, non-linear data and how the differences for those look.

Example 3.3.7 Quadratic Data. Consider Table 3.3.8. The first differences (what we calculated above) are not the same. Thus this data is not linear.

However, the first differences increase in a suspiciously simple pattern (odd numbers). Checking the second differences (the differences of the 1st differences) we see a linear pattern. This turns out to be the pattern for all **quadratic** data. ▲

Table 3.3.8 Quadratic Data

n	n^2	1st difference	2nd difference
1	1		
2	4	4-1=3	
3	9	9-4=5	5-3=2
4	16	16-9=7	7-5=2
5	25	25-16=9	9-7=2
6	36	36-25=11	11-9=2

Definition 3.3.9 Quadratic Relation. A relation is **quadratic** if and only if the second differences are constant. ◆

Example 3.3.10 Exponential Data. Consider Table 3.3.11. The differences are not the same nor do they show the same pattern as quadratics. However, there is a pattern in the differences. Notice that the differences are exactly equal to the original data. This means that the rate of increase is determined by the current scale. In other words, the bigger it is, the faster it grows. This is the pattern of data that varies exponentially. ▲

Table 3.3.11 Exponential Data

n	2^n	Difference
1	2	
2	4	4-2=2
3	8	8-4=4
4	16	16-8=8
5	32	32-16=16
6	64	64-32=32

The next example illustrates that the differences for exponential data are not always exactly equal to the data.

Example 3.3.12 Exponential Data Differences. Consider Table 3.3.13. The differences are not exactly equal to the original numbers. However, notice that $6 = 2 \cdot 3$, $18 = 2 \cdot 9$, and $54 = 2 \cdot 27$. The differences are double the original numbers. In general for exponential data the differences will be the original data scaled by some number.

Happily there is an easier way to determine that data is exponential shown in the next section. ▲

Table 3.3.13 Exponential Data with Scale

n	3^n	Difference
1	3	
2	9	9-3=6
3	27	27-9=18
4	81	81-27=54
5	243	243-81=162
6	729	729-243=486

Definition 3.3.14 Exponential Relation. A relation is **exponential** if and only if the differences are a multiple of the original values, that is the rate is proportional to the value. ◆

3.3.2 Quotients

The previous section analyzed change as the difference (subtraction) of consecutive numbers (salaries in these examples). This section analyzes change using the percent increase for each pair of consecutive numbers.

Example 3.3.15 Salary Percent Incease. We will first calculate the percent increase of salary each year for Tien and Vasya. Because salary numbers are exact, we will not use significant digits. Rather we will round to the nearest percent. This is in Table 3.3.16 and Table 3.3.17.

Notice that for Tien the percent increase is the same each year. It is 5%. For Vasya, the percent increase is not the same each year. How does the percent increase change for her? ▲

Table 3.3.16 Percent Increase for Tien

Year	Ratio		Increase
2018	\$55,050.80/\$52,429.33	=1.05	5%
2019	\$57,803.34/\$55,050.80	=1.05	5%
2020	\$60,693.50/\$57,803.34	=1.05	5%
2021	\$63,728.18/\$60,693.50	=1.05	5%

Table 3.3.17 Percent Increase for Vasya

Year	Ratio		Increase
2018	\$67,347.23/\$62,347.23	=1.08	8%
2019	\$72,347.23/\$67,347.23	=1.07	7%
2020	\$77,347.23/\$72,347.23	=1.07	7%
2021	\$82,347.23/\$77,347.23	=1.06	6%

Definition 3.3.18 Exponential. A relation is **exponential** if and only if the percent increase is constant. ◆

Example 3.3.19 The table below gives an amount of caffeine in the blood stream. This data is exponential with a ratio of 0.87. The ratio in this example means there is a 13% decrease per hour of caffeine in the blood stream. This is in contrast to the previous example which was an increasing exponential.

Hour	Caffeine	Ratio
0	95 mg	
1	83 mg	$83/95 \approx 0.87$
2	72 mg	$72/83 \approx 0.87$
3	63 mg	$63/72 \approx 0.87$
4	55 mg	$55/63 \approx 0.87$
5	48 mg	$48/55 \approx 0.87$
6	41 mg	$41/48 \approx 0.87$
7	36 mg	$36/41 \approx 0.87$
8	31 mg	$31/41 \approx 0.87$

▲

Although Definition 3.3.14 and Definition 3.3.18 are phrased differently they both accurately describe exponential relations. Generally it is easier to test if data is exponential by testing the ratios of terms rather than the differences. Table 3.3.20 shows an example of analyzing data using both differences and ratios. Notice that the differences are a scaled version of the original data (scaled by 1/3). The ratio from the quotient is 4/3 which gives about a 33% increase. For the curious the data was generate by $5\left(\frac{4}{3}\right)^n$.

Table 3.3.20 Exponential Data 2 Ways

Data	Difference	Ratio
$\frac{20}{3}$		
$\frac{80}{9}$	$\frac{20}{9} = \frac{1}{3} \cdot \frac{20}{3}$	$\frac{4}{3} \approx 1.33$
$\frac{320}{27}$	$\frac{80}{27} = \frac{1}{3} \cdot \frac{80}{9}$	$\frac{4}{3} \approx 1.33$
$\frac{1280}{81}$	$\frac{320}{81} = \frac{1}{3} \cdot \frac{320}{27}$	$\frac{4}{3} \approx 1.33$
$\frac{5120}{243}$	$\frac{1280}{243} = \frac{1}{3} \cdot \frac{1280}{81}$	$\frac{4}{3} \approx 1.33$
$\frac{20480}{729}$	$\frac{5120}{729} = \frac{1}{3} \cdot \frac{5120}{243}$	$\frac{4}{3} \approx 1.33$

Checkpoint 3.3.21 Determine whether each person's salaries followed a linear or an exponential growth pattern.

Year	Benicio	Craig	Palesa
2019	63998.02	55075	67389
2020	65862.04	55625.75	68736.78
2021	67726.06	56182.01	70111.52
2022	69590.08	56743.83	71513.75
2023	71454.1	57311.27	72944.02
2024	73318.12	57884.38	74402.9

Benicio:

1. linear
2. exponential

Craig:

1. linear
2. exponential

Palesa:

1. linear
2. exponential

3.3.3 Extrapolation

In Example 3.2.6 we found a value between two entries in a table. That is interpolation. In other cases we want to find a value past the end of a table. This is called **extrapolation**.

Example 3.3.22 Extrapolation from a Table. Based on Table 3.3.1 what do we expect Tien's salary to be in 2022, 2025?

From Example 3.3.15 we know each entry is 1.05 times the previous year's salary. Because that was the pattern every year, we might safely suppose it will occur again. Thus we expect his 2022 salary to be $1.05 \cdot \$73,773.33 \approx \$77,462.00$.

To extrapolate to 2025 we repeat this process for 2023, 2024, and 2025.

$$1.05 \cdot \$77,462.00 \approx \$81,335.10.$$
$$1.05 \cdot \$81,335.10 \approx \$85,401.85.$$
$$1.05 \cdot \$85,401.85 \approx \$89,671.94.$$

If his raises continue at the same rate he will have a salary of \$89,671.94 in 2025. ▲

Example 3.3.23 Based on Table 3.2.21 what do we expect Vasya's salary to be in 2025, 2028?

From Example 3.2.20 we know that Vasya has received a \$5,000 raise each year. Because that has been the pattern, we might safely suppose it will continue. Thus we expect her 2025 salary to be

$$\$97,347.23 + \$5,000.00 = \$102,347.23.$$

To calculate her expected 2028 salary we note that is 4 years after 2024, so she should have four raises of \$5,000 each. Her expected salary will be

$$\$97,347.23 + 4(\$5,000.00) = \$117,347.23.$$

▲

Notice that in both of these examples we needed to know the growth rate (i.e., exponential or linear respectively) before we could extrapolate.

Checkpoint 3.3.24 Determine whether Xavier's salary follows a linear or an exponential growth pattern. Then extrapolate to determine the expected salary in 2028.

Year	Xavier
2019	43993
2020	45312.79
2021	46672.17
2022	48072.34
2023	49514.51
2024	50999.94

The salary growth is

1. linear
2. exponential

The salary in 2028 is expected to be _____

3.3.4 Exercises

Identifying Rates. Determine from the tables of data what the rate of growth is.

1. Determine if the following table represents a linear relation.

x	0	1	2	3	4	5
y	0.1	-2.8	-5.7	-8.6	-11.5	-14.4

1. Yes
2. No

2. Determine if the following table represents a linear relation.

x	0	1	2	3	4	5
y	1.2	2.8	7.6	15.6	26.8	41.2

1. Yes
2. No

3. Determine if the following table represents a linear relation.

x	0	1	2	3	4	5
y	3.9	4.9	6.9	10.9	18.9	34.9

1. Yes
2. No

4. Determine if the following table represents a linear relation.

x	1	2	3	4	5	6
y	684	171	76	42.75	27.36	19

1. Yes
2. No

5. For each table below, could the table represent a function that is linear, exponential, or neither?

x	1	2	3	4
f(x)	70	49	34.3	24.01

f(x) is

1. Exponential
2. Linear
3. Neither

x	1	2	3	4
g(x)	70	40	10	-20

g(x) is

1. Exponential
2. Linear
3. Neither

x	1	2	3	4
h(x)	40	-11	-55.7	-95.99

h(x) is

1. Exponential
2. Linear
3. Neither

6. Which one of these tables shows a quadratic relationship in the data?

1.

x	0	1	2	3	4	5
y	4	7	10	13	16	19

2.

x	0	1	2	3	4	5
y	6	18	54	162	486	1458

3.

x	0	1	2	3	4	5
y	2	5	11	20	32	47

7. Determine if the relation in each table is linear, exponential, or quadratic.

x	0	1	2	3	4	5
y	4	8	16	32	64	128

is:

1. Linear
2. Exponential
3. Quadratic

x	0	1	2	3	4	5
y	2	5	11	20	32	47

is:

1. Linear
2. Exponential
3. Quadratic

x	0	1	2	3	4	5
y	7	9	11	13	15	17

is:

1. Linear
2. Exponential
3. Quadratic

Extrapolating. Identify the type of relation and extrapolate to write a few more elements.

8. For the following sequence, state the next three terms.
6, 18, 54, 162, ___, ___, ___
What type of sequence is this?

1. linear
2. quadratic
3. exponential

4. other

9. For the following sequence, state the next three terms.

4, 6, 9, 13, ___, ___, ___

What type of sequence is this?

1. arithmetic (linear)
2. quadratic
3. geometric (exponential)
4. other

10. Identify the pattern in each table as linear, exponential, or quadratic. Extrapolate to fill in two more entries.

x	y
0	3
1	10
2	17
3	24
4	__
5	__
6	45

is:

1. Linear
2. Exponential
3. Quadratic

x	y
0	9
1	18
2	36
3	72
4	__
5	__
6	576

is:

1. Linear
2. Exponential
3. Quadratic

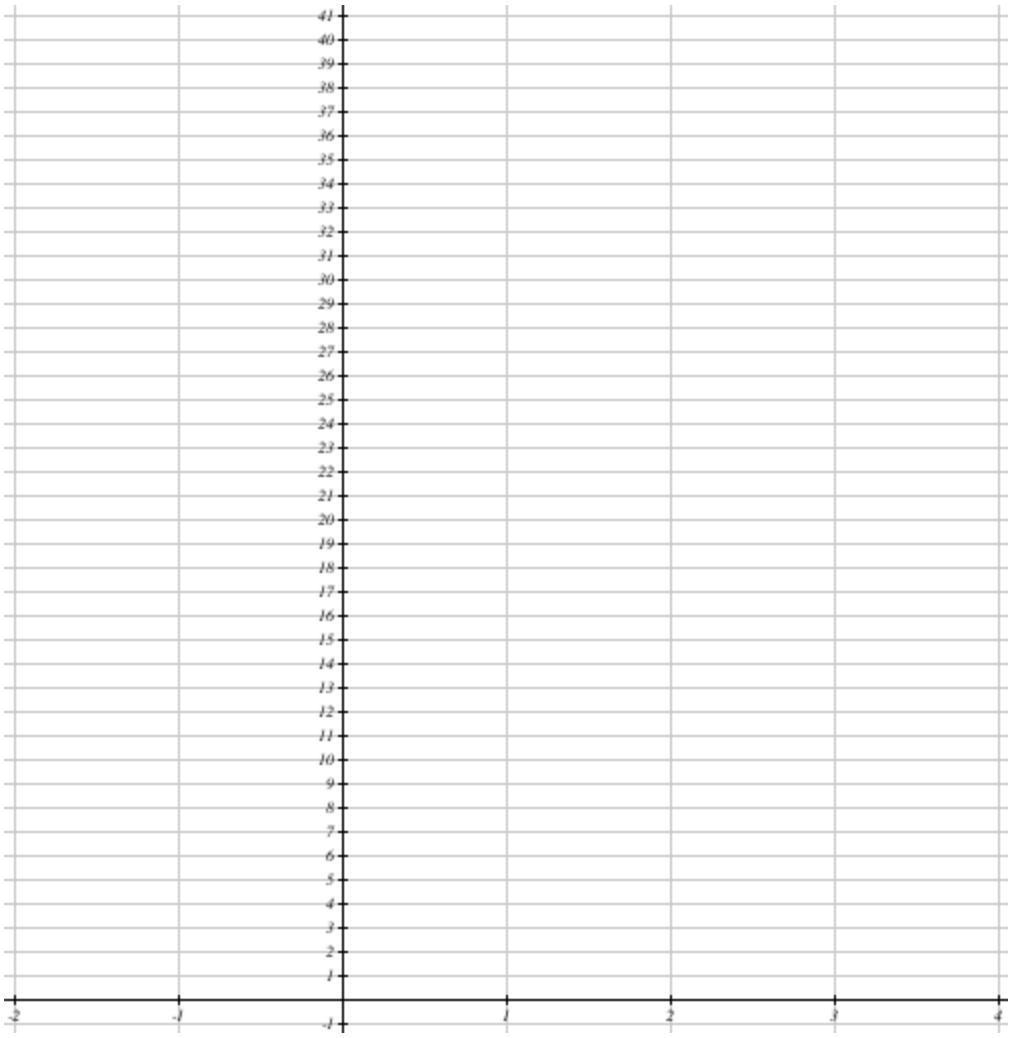
x	y
0	4
1	5
2	7
3	10
4	__
5	__
6	25

is:

1. Linear
2. Exponential
3. Quadratic

11. Directions: Graph each set of values (just the points), and determine whether the function is linear, quadratic, or exponential.

x	-1	0	1	2	3
y	8	4	8	20	40



The function type is:

1. Linear
2. Quadratic
3. Exponential

I know this, because the graph/table shows that

1. the rate of change is constant

2. the change of the change is constant
3. each subseqent value is multiplied by a constant number

12. For each scenario, identify the appropriate growth model that describes how it's changing.

1. The amount of pollutants in the lake has been increasing by 4 milligrams per Liter each year
2. Tuition is currently \$2,000 a quarter has been growing by 7% a year
3. The number of arrests grew for several years, but now has been decreasing
4. The number of new polio cases has been cut in half each year due to vaccination efforts

1. Linear
2. Exponential
3. Neither

3.4 Variation

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)
- Identify data varrying directly or indirectly (critical thinking)
- Solve linear, rational, quadratic, and exponential equations and formulas (skill)

Section 3.3 presents how to determine the rate at which one variable grows with respect to another. In all of those cases we considered only one variable changing. However, many models have more than one variable or parameter, and we wish to analyze the impact each one has on the result. This section will also present one more type of rate.

3.4.1 Analyzing Models with Multiple Variables

This first example illustrates how a result can be linear with respect to one or more variables and quadratic (or other relationships) with another.

Example 3.4.1 Review Model 1.3.3. Consider how lift changes with respect to each of the parameters.

Consider ρ the density of air. If the other parameters remain constant (only density changes), then the model looks like

$$\begin{aligned} L &= \frac{1}{2}\rho s C_L v^2 \\ &= \left(\frac{1}{2} s C_L v^2\right)\rho \\ &= k\rho. \end{aligned}$$

All we used is commutativity of multiplication. We can see that lift (L) changes linearly with respect to air density (ρ).

Consider s the surface area of the airfoil (wing or propeller). If the other parameters remain constant (only surface area changes), then the model looks like

$$\begin{aligned} L &= \frac{1}{2}\rho s C_L v^2 \\ &= \left(\frac{1}{2}\rho C_L v^2\right) s \\ &= ks. \end{aligned}$$

We can see that lift (L) changes linearly with respect to surface area (s).

Consider C_L the coefficient of lift of the airfoil. If the other parameters remain constant, then the model looks like

$$\begin{aligned} L &= \frac{1}{2}\rho s C_L v^2 \\ &= \left(\frac{1}{2}\rho s v^2\right) C_L \\ &= kC_L. \end{aligned}$$

We can see that lift (L) changes linearly with respect to the coefficient of lift (C_L).

Consider v the velocity. If the other parameters remain constant, then the model looks like

$$\begin{aligned} L &= \frac{1}{2}\rho s C_L v^2 \\ &= \left(\frac{1}{2}\rho s C_L\right) v^2 \\ &= kv^2. \end{aligned}$$

We can see that lift (L) changes quadratically with respect to velocity (v^2).

How can we apply this knowledge? Lift changes linearly with respect to all parameters except for velocity. If greater lift (to handle greater weight) is needed, velocity provides bigger bang for our buck than any other change. ▲

This second example has us look at the impact on more than just one variable. It also introduces a new relationship.

Example 3.4.2 Review Model 1.3.5.

First consider the impact temperature (T) has on pressure (P). If the other parameters remain constant, then the model looks like

$$\begin{aligned} PV &= nRT. && \\ P &= \frac{nRT}{V}. && \text{Divide to isolate } P \\ P &= \frac{nR}{V}T. && \text{Use commutativity} \\ P &= kT. && \end{aligned}$$

Pressure (P) grows linearly with respect to temperature (T).

Notice that we can write that last equation as $\frac{1}{k}P = T$ which means we can also make the reverse statement: temperature increases linearly with pressure.

Next consider the impact temperature (T) has on volume (V). If the other parameters remain constant, then the model looks like

$$\begin{aligned} PV &= nRT. \\ V &= \frac{nRT}{P}. \end{aligned}$$

$$V = \frac{nR}{P}T.$$
$$V = kT.$$

Volume (V) also grows linearly with respect to temperature (T).

Finally consider the impact volume (V) has on pressure (P). If the other parameters remain constant, then the model looks like

$$PV = nRT.$$
$$P = \frac{nRT}{V}.$$
$$P = (nRt)\frac{1}{V}.$$
$$P = k \cdot \frac{1}{V}.$$

This pattern is not linear (nor quadratic, nor exponential). If volume is increased the right hand side will decrease (dividing by a larger number). This means that pressure will decrease. Conversely if volume is decreased the right hand side will increase (divide by a smaller number). This means that pressure will increase. ▲

Definition 3.4.3 Vary Directly. For two quantities a and b, if increasing b increases a, then a is said to **vary directly** with b. ◆

Definition 3.4.4 Vary Indirectly. For two quantities a and b, if increasing b decreases a, then a is said to **vary indirectly** with b. ◆

Example 3.2.31 illustrated a model where the increase in one variable caused a decrease in the other. We used an experiment (plugging in numbers) to discover the inverse relationship.

If two quantities vary directly, the model may be linear, quadratic, or exponential. If two quantities vary inversely, the model may be neither linear nor quadratic. In contrast the next example illustrates two quantities can vary inversely, and the model is exponential.

Example 3.4.5 Decreasing Exponential. Review the data in the table below. First, notice that a decreases as n increases which means a varies inversely with n. Second, note that the data is exponential with a ratio of 1/2. All exponential models that are decreasing have this property.

n	$a = \frac{1}{2^n}$	Ratio
1	1/2	
2	1/4	1/2
3	1/8	1/2
4	1/16	1/2
5	1/32	1/2
6	1/64	1/2

▲

3.4.2 More Model Usage

In the section above and in Section 3.3 we determined growth rates (e.g., linear, quadratic exponential, and direct and inverse variation) by generating data and checking the consecutive differences and/or the consecutive ratios. In this section we learn a method to directly calculate the relationship.

Model 3.4.6 *Pressure is a measure of the amount of force spread over an area.*

$$P = \frac{F}{A}$$

where

- *P is pressure,*
- *F is the force in units of pounds (lbs) or Newtons (N),*
- *A is the area in units of square inches, or meters, or similar.*

Example 3.4.7 Walking on snow (or mud) is difficult because our feet tend to puncture the snow (high pressure applied to low strength snow). We can use snowshoes to avoid this problem: they reduce pressure with minimal increase in force (weight). This is a science model with measurements, so we will round using significant digits.

Determine the pressure Guido exerts on the snow without and with snow shoes. Suppose Guido weighs 172 lbs.

Without snowshoes each foot has an area of $22\,\text{in}^2$. What is the pressure he exerts on the snow?

$$\begin{aligned} P &= \frac{172 \text{ lbs}}{2 \cdot 22 \text{ in}^2} \\ &= \frac{172 \text{ lbs}}{44 \text{ in}^2} \\ &\approx 3.909090 \text{ lbs/in}^2 \\ &\approx 3.9 \text{ psi.} \end{aligned}$$

If he wears snowshoes that have a surface area of $144\,\text{in}^2$, what is the pressure?

$$\begin{aligned} P &= \frac{172 \text{ lbs}}{2 \cdot 144 \text{ in}^2} \\ &= \frac{172 \text{ lbs}}{288 \text{ in}^2} \\ &\approx 0.597222 \text{ lbs/in}^2 \\ &\approx 0.60 \text{ psi.} \end{aligned}$$

▲

The next example illustrates how we can construct another model when we are interested in only some of the variables.

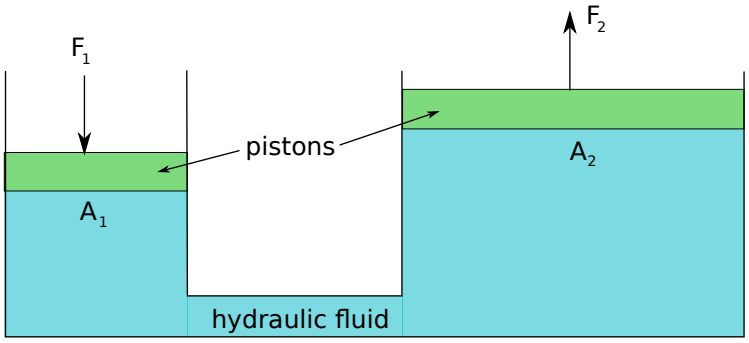


Figure 3.4.8 Hydraulic Press

Model 3.4.9 Hydraulics. *Consider the situation in Figure 3.4.8. The pressure exerted on a fluid by a piston is the ratio of the force exerted and the area of the piston. On the left that is $P = \frac{F_1}{A_1}$. The same is true on the right $P = \frac{F_2}{A_2}$. Because the hydraulic fluid is contiguous the pressure is the same on both sides. Thus*

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}.$$

Example 3.4.10 If the left piston has area 16 cm^2, and 5.0 N of force is exerted, what is the force exerted on the second piston if it has area 25 cm^2? This is a science model with measurements, so we will round

using significant digits.

We notice this model is a proportion. We can use the relationship

$$\begin{aligned}\frac{5.0\text{ N}}{16\text{ cm}^2} &= \frac{F_2}{25\text{ cm}^2}.\\ \frac{5.0\text{ N}}{16\text{ cm}^2}\cdot(25\text{ cm}^2) &= F_2.\\ 7.8125 &= F_2.\\ 7.8\text{ N} &\approx F_2.\end{aligned}$$

▲

The next example illustrates how we can determine how one variable varies with respect to another by direct calculation. Specifically, consider how changing the size of the second piston affects the resulting force.

Example 3.4.11 If the second piston's area is changed from 25 cm^2 to 36 cm^2, what is the increase or decrease in force?

Solution. To determine if it increases or decreases we should calculate the result for the second piston size. We set up the same problem as above.

$$\begin{aligned}\frac{5.0\text{ N}}{16\text{ cm}^2} &= \frac{F_2}{36\text{ cm}^2}.\\ \frac{5.0\text{ N}}{16\text{ cm}^2}\cdot(36\text{ cm}^2) &= F_2.\\ 11.25 &= F_2.\\ 11\text{ N} &\approx F_2.\end{aligned}$$

The piston is larger and the force is also larger. Thus we know that force and area vary directly. The equation also shows us that they increase linearly, because all terms are linear. In particular $F_2 = A_2\left(\frac{F_1}{A_1}\right)$ is in the form of a line with ratio F_1/A_1 and no shift. ▲

Model 1.3.5 gives two forms of the ideal gas law. The second is produced from the first in the same way as the hydraulics model is produced from the pressure definition. To illustrate consider the case of the air/water mixture in a pressure cooker. First recall the ideal gas law $PV = nRT$. The pressure cooker changes the temperature of the gas, but the volume, number of molecules and gas constant do not change. Thus we have $P_1V = nRT_1$ and $P_2V = nRT_2$. In both cases we can solve for the shared variables.

$$\begin{aligned}P_1V &= nRT_1\\ P_1V\cdot\frac{1}{VT_1} &= nRT_1\cdot\frac{1}{VT_1}\\ \frac{P_1}{T_1} &= \frac{nR}{V}\end{aligned}$$

Similarly $\frac{P_2}{T_2} = \frac{nR}{V}$. Because both expressions equal nR/V we can set them equal giving $\frac{P_1}{T_1} = \frac{P_2}{T_2}$ which is the other model.

The example above produces a second form of the ideal gas law. The next example produces a second form of Ohm's Law. In many applications the voltage is fixed. For example it might be 12V from a car battery. Thus if we are considering current and resistance in two places we have $V = I_1R_1$ and $V = I_2R_2$. Because voltage is the same (same source of electricity) we have $I_1R_1 = I_2R_2$.

Example 3.4.12 If the current is 3 amps when the resistance is 8 Ohms, what will the current be when the resistance is 6 Ohms? 4 Ohms?

Solution. We can write

$$\begin{aligned}I_1R_1 &= I_2R_2\\ (3\text{ amps})(8\Omega) &= I_2(6\Omega)\end{aligned}$$

$$\frac{(3 \text{ amps})(8\Omega)}{6\Omega} = I_2$$
$$4 \text{ amps} = I_2.$$

The second case can be written

$$(3 \text{ amps})(8\Omega) = I_2(4\Omega)$$
$$\frac{(3 \text{ amps})(8\Omega)}{4\Omega} = I_2$$
$$6 \text{ amps} = I_2.$$

Notice as the resistance decreased, the current increased. Current varies inversely with resistance. ▲

3.4.3 Limitations of Models

In Example 3.4.2 we solved the model equation to see how one value changes with respect to another. We considered how temperature impacts pressure if the other properties do not change. We also considered how temperature impacts volume if the other properties do not change. We might ask how how volume impacts temperature if the other properties do not change.

We will not however, because this is impossible. If volume is reduced (e.g., we press down on the handle of a tire pump and do not let the air out), then the pressure will increase. There is no way around that. The increase of pressure will then result in an increase of temperature. This still matches the model we have.

When using a model, mathematics is used to understand what is meant, but we must understand some of the background so we do not draw false conclusions.

3.4.4 Exercises

Distinguish Direct and Indirect Variation. Based on the model, determine how each parameter (variable) affects the others.

1.



$$v = \left(\frac{2D}{\rho C A}\right)^{1/2}$$

- D is the skydiver's weight
- ρ is the density of the air
- C is the skydiver's coefficient of drag
- A is the skydiver's ground-facing surface area

(a) Increasing D will

1. increase velocity
2. decrease velocity

(b) Increasing ρ will

1. increase velocity

2. decrease velocity

(c) Decreasing C will

1. increase velocity
2. decrease velocity

(d) Decreasing A will

1. decrease velocity
2. increase velocity

2. Power, voltage, and current are related by $P = I \cdot V$ where
P is the power in watts,
I is the current in amps,
and V is the voltage in volts.
An decrease voltage will

1. decrease power
2. increase power

An increase current will

1. decrease power
2. increase power

Interpret. Use the definitions of the terms to determine what will happen.

3. For the following exercise, assume the constant k is positive.
p varies directly with q. Describe what happens to the value of p as q decreases.

1. increases
2. undeterminable
3. no change
4. decreases

4. For the following exercise, assume the constant k is positive.
W varies inversely with n. Describe what happens to the value of W as n decreases.

1. decreases
2. undeterminable
3. no change
4. increases

Terminology. Answer these questions based on the stated variation.

5. S varies directly as p and q. If $p = 6$ and $q = 7$ then $S = 277.2$. Find the constant of proportionality.
$k =$ _____

6. Write the equation representing the relationship, use k for the constant of variation.
e varies directly as t

1. e t = k

2. e / t = k

7. Write the equation representing the relationship, use k for the constant of variation.
f is inversely proportional to r

1. f / r = k

2. f r = k

8. Write a function describing the relationship of the given variables.
V varies directly with the square of t and when $t = 6$, $V = 540$
$V =$_____

9. Write a function describing the relationship of the given variables.
y varies inversely with the square of x and when $x = 3$, $y = 5$
$y =$_____

10. Write a function describing the relationship of the given variables.
A varies directly with the square root of r and when $r = 16$, $A = 56$
$A =$_____

Applications. Use a model based on the stated variation to answer these questions.

11. The force F (in pounds) needed on a wrench handle to loosen a certain bolt varies inversely with the length L (in inches) of the handle. A force of 40. pounds is needed when the handle is 7.0 inches long. If a person needs 15 pounds of force to loosen the bolt, estimate the length of the wrench handle. Calculate using significant digits.
_____ inches

12. The electrical current, in amperes, in a circuit varies directly as the voltage. When 27 volts are applied, the current is 9 amperes. What is the current when 60 volts are applied?
_____ amperes

13. The number of hours required to build a fence is inversely proportional to the number of people working on the fence. If it takes 4 people 44 hours to complete the fence, then how long will it take 11 people to build the fence?
(Round the answer to 2 decimal places if needed)
____ hours.

14. The capacitive reactance, X, in a circuit varies inversely as the frequency, f, of the applied voltage. If the reactance is 624 ohms when the frequency is 68.9 hertz, find the reactance when the frequency is 46.2.
_____ Ω

15. The loudness, L, of a sound (measured in decibels, dB) is inversely proportional to the square of the distance, d, from the source of the sound.
When a person 9.0 feet from a jetski, it is 60.0 decibels loud. How loud is the jetski when the person is 50 feet away?
Round using the rules of significant figures.
_____ dB

16. Hooke's law states that the distance that a spring is stretched by hanging object varies directly as the mass of the object. If the distance is 100.0 cm when the mass is 15.0 kg, what is the distance when the mass is 8.00 kg?
Round using the rules of significant figures.
_____ cm

17. The volume of a gas varies inversely as the pressure upon it. The volume of a gas is 800 cm^3 under a pressure of 8 kg/^2. What will be its volume under a pressure of 640 kg/cm^2?
Round your answer to two significant figures.
_____ cm^3

18. The wavelength of a radio wave varies inversely as its frequency. A wave with a frequency of 1200 kilohertz has a length of 300 meters. What is the length of a wave with a frequency of 450 kilohertz?
_____ meters

19. The velocity v of a falling object varies directly with the time t of the fall. If after 2.00 seconds, the velocity of an object is 64.0 feet per second, what is the velocity after 11.0 seconds?
_____ feet per second
Round using significant figures.

20. The weight of an object above the surface of Earth varies inversely with the square of distance from the center of Earth. If an object weighs 90.00 pounds when it is 3960. miles from Earth's center, what would the same object weigh when it is 4,020. miles from Earth's center?
_____ pounds
Round using significant figures.

21. Newton's Law of Gravitation says that two objects with masses m_1 and m_2 attract each other with a force F that is jointly proportional to their masses and inversly proportional to the square of the distance r between the objects. Newton discovered the constant of proportionality is 6.67×10^{-11}.
In a small laboratory experiment, two 600. kg masses are separated by 0.800 meters. What would the gravitational force between the objects be?
Force = _____ Newtons

3.5 Rational Expressions

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)

This section presents algebra needed to work with models involving more complex, rational (fractional) expressions; and it presents how to answer questions requiring adding rates.

3.5.1 Re-arranging Rational Expressions

In previous sections many of the models involving rational (fractional) expressions could be set up so we solved for a variable in the numerator. In this section we look at multiple examples in which we must solve for a variable in the denominator.

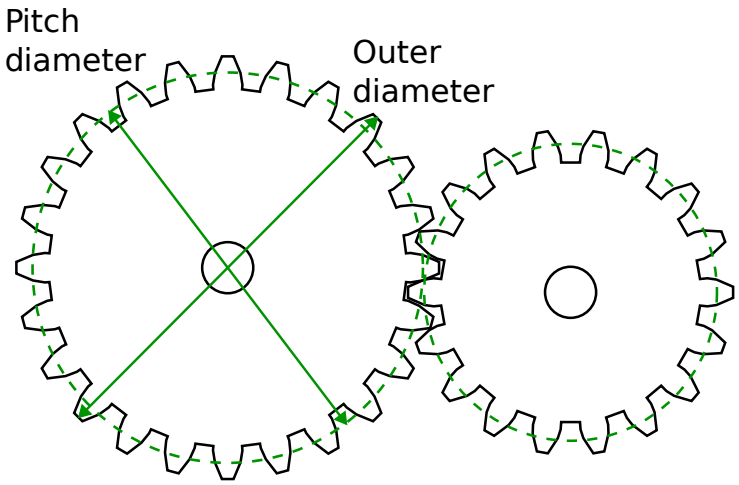


Figure 3.5.1 Gear Measurements

Model 3.5.2 Gear Design. *For ANSI standard gears there is a relationship between the number of teeth, and diameters of the gear.*

$$D_p = \frac{D_o N}{N+2}$$

where

- N *is the total number of teeth*
- D_o *the diameter of the outside of the gear, and*
- D_p *is the pitch diameter.*

Outer diameter is the diameter of a circle that contains the tips of the teeth. Pitch diameter is the diameter of a circle such that this gear touches the other gear at this distance. It is illustrated as the dashed circles in the figure above.

Example 3.5.3 Suppose we know the outer diameter (17/8") and pitch diameter (2") needed for a gear. What are the steps to solve for the number of teeth? The number of teeth must end up an integer (half a gear tooth is just a broken gear). Thus there will be no rounding. If the calculation gives a non-integer result, we must change something in the design.

First we convert the outer diameter to decimal: $17/8 = 2.125$. The model is

$$D_p = \frac{D_o N}{N+2}.$$
$$2 \text{ in} = \frac{(2.125 \text{ in})N}{N+2}.$$

First, we multiply to eliminate the fraction.

$$(2 \text{ in}) \cdot (N+2) = \frac{(2.125 \text{ in})N}{N+2} \cdot (N+2).$$

Second, we distribute so we can access the variable.

$$(2 \text{ in})N + (4 \text{ in}) = (2.125 \text{ in})N$$

Next, we need to collect N on one side.

$$-(2 \text{ in})N + (2 \text{ in})N + 4 \text{ in} = -(2 \text{ in})N + (2.125 \text{ in})N.$$

$$\begin{aligned} 4 \text{ in} &= (0.125 \text{ in})N \\ \frac{4 \text{ in}}{0.125 \text{ in}} &= \frac{(0.125 \text{ in})N}{0.125 \text{ in}}. \\ 32 &= N. \end{aligned}$$

Thus this gear will have 32 teeth (total). ▲

Example 3.5.4 Formula for Number of Teeth. If we are going to perform this calculation regularly, we can solve the equation for N.

$$\begin{aligned} D_p &= \frac{D_o N}{N+2}. \\ D_p \cdot (N+2) &= \frac{D_o N}{N+2} \cdot (N+2). \\ D_p(N+2) &= D_o N. \\ D_p N + 2D_p &= D_o N. \\ D_p N - D_o N &= -2D_p. \\ (D_p - D_o)N &= -2D_p. \\ \frac{(D_p - D_o)N}{D_p - D_o} &= \frac{-2D_p}{D_p - D_o}. \\ N &= \frac{-2D_p}{D_p - D_o}. \end{aligned}$$

Notice we needed to collect the terms with N. This required distributing (third line), collecting on one side, then factoring. ▲



Standalone
In Context

Figure 3.5.5 Pitch Diamter

A problem with similar algebra is the Model 1.3.5 when we are solving for temperature. The next example illustrates the necessary algebra.

Example 3.5.6 Suppose the conditions for a tire are $P_1 = 3\bar{0}$ psi at a temperature of $T = 52°$ F. At what temperature will the pressure drop below the safe value of 28 psi? Because our gauges are not more accurate we will round to units.

We can use the ideal gas law version below. Note that the volume does not change (the tire size does not change).

$$\frac{P_1 V_1}{T_1 + 460°} = \frac{P_2 V_2}{T_2 + 460°}$$

Because the volume does not change, $V_2 = V_1$.

$$\begin{aligned}
\frac{(30\text{ psi})V_1}{52^\circ + 460^\circ} &= \frac{(28\text{ psi})V_1}{T_2 + 460^\circ} \\
\frac{(30\text{ psi})V_1}{512^\circ} \cdot \frac{1}{V_1} &= \frac{(28\text{ psi})V_1}{T_2 + 460^\circ} \cdot \frac{1}{V_1} \\
\frac{30\text{ psi}}{512^\circ} &= \frac{28\text{ psi}}{T_2 + 460^\circ}
\end{aligned}$$

Recall a proportion works both directions.

$$\begin{aligned}
\frac{512^\circ}{30\text{ psi}} &= \frac{T_2 + 460^\circ}{28\text{ psi}} \\
\frac{512^\circ}{30\text{ psi}} \cdot (28\text{ psi}) &= T_2 + 460^\circ \\
477.866667^\circ &\approx T_2 + 460^\circ \\
477.866667^\circ - 460^\circ &\approx T_2 \\
17.8666666^\circ &\approx T_2. \\
18^\circ &\approx T_2.
\end{aligned}$$

▲

Checkpoint 3.5.7 A known volume of ideal gas is contained in a cylinder fitted with a piston and held at constant pressure. If the temperature changes from 25°C to 200.°C, by what factor does the volume change? Round your answer using the rules for working with significant figures.

For this problem you should know that when pressure is held constant, gasses follow Charles' Law, which states $\frac{V_1}{T_1 + 273} = \frac{V_2}{T_2 + 273}$, where V_1 and T_1 is the initial volume and temperature and V_2 and T_2 is the new volume and temperature.

The volume

1. decreases
2. increases

by a factor of _____

Some rational expressions have the variable in both numerator and denominator. These are handled exactly the same: we can still multiply to eliminate the fractions.

Example 3.5.8 The following expression comes from an equilibrium problem in chemistry. Here we need to convert this rational expression into a polynomial.

$$\frac{[HBr]^2}{[H_2][Br_2]} = \frac{(2x)^2}{(0.700 - x)(0.500 - x)} = 64.0.$$

$$\frac{(2x)^2}{(0.700 - x)(0.500 - x)} = 64.0.$$

Multiply to eliminate the fraction.

$$\begin{aligned}
\frac{(2x)^2}{(0.700 - x)(0.500 - x)} \cdot (0.700 - x)(0.500 - x) &= 64.0 \cdot (0.700 - x)(0.500 - x). \\
(2x)^2 &= 64.0 \cdot (0.700 - x)(0.500 - x).
\end{aligned}$$

$$4x^2 = 64.0(0.350 - 1.200x + x^2).$$
$$4x^2 = 22.4 - 76.8x + 64.0x^2.$$
$$0 = 22.4 - 76.8x + 60.0x^2.$$

Later (Subsection 5.2.2), we will learn to solve this equation. ▲

The final example in this section presents a type of problem involving adding multiple fractions. This is needed for the resistors in parallel problem and is connected to the next section on additive rates.

Model 3.5.9 Parallel Resistors. *When two resistors are in parallel as shown in Figure 3.5.10 then the resulting resistance is given by*

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If there are more than two resistors, the model is expanded by adding an additional $1/R$ *term for each resistor.*

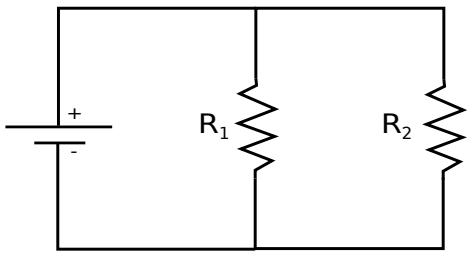


Figure 3.5.10 Resistors in Parallel

The primary algebra technique required is to obtain common denominators.

Example 3.5.11 Parallel Resistance. Calculate the resulting resistance when one resistor is 4 Ohms ($R_1 = 4$) and the other is 12 Ohms ($R_2 = 12$).

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2}. \\
\frac{1}{R} &= \frac{1}{4} + \frac{1}{12}. \\
\frac{1}{R} &= \frac{3}{3} \cdot \frac{1}{4} + \frac{1}{12}. \text{ Common denominator} \\
\frac{1}{R} &= \frac{3}{12} + \frac{1}{12}. \\
\frac{1}{R} &= \frac{4}{12} \\
&= \frac{1}{3}. \\
\frac{1}{R} \cdot 3R &= \frac{1}{3} \cdot 3R \text{ Clear both denominators} \\
3 &= R.
\end{aligned}$$

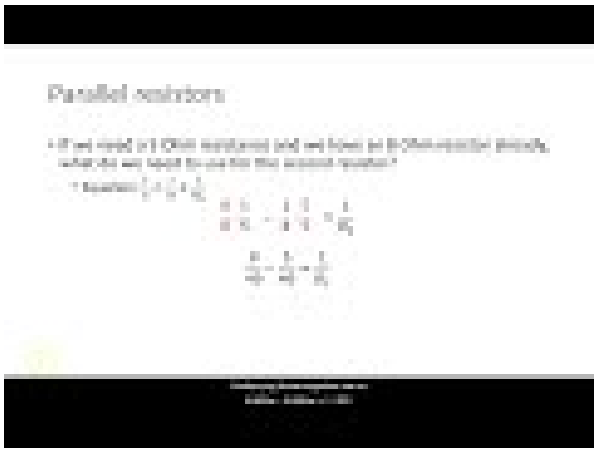
Note the need for a common denominator in the third line. The final step is our now frequently used clearing of denominators (i.e., 'cross multiplication'). ▲

Example 3.5.12 Parallel Resistance Solving. In the previous example we knew the two resistors and calculated the resulting resistance. In other cases we know how much resistance we need and one of the resistors. We must calculate the resistance for the other resistor.

If we need a 5 Ohm resistance and have an 8 Ohm resistor already, what do we add as the second resistor? Resistor measurements are commonly accurate to ±5%. For this problem that allows us to round to units. This is because 5% of 5 is much less than one.

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2}. \\
\frac{1}{5} &= \frac{1}{8} + \frac{1}{R_2}. \\
\frac{1}{5} - \frac{1}{8} &= \frac{1}{R_2}. \\
0.2 - 0.125 &= \frac{1}{R_2} \\
0.075 &= \frac{1}{R_2} \\
0.075R_2 &= 1 \\
R_2 &= \frac{1}{0.075} \\
&\approx 13.333333 \\
&\approx 13.
\end{aligned}$$

▲





Standalone
In Context

Figure 3.5.13 Parallel Resistance Solving

Checkpoint 3.5.14 If two resistors are arranged in parallel one is 6.0Ω and the other is 8.0Ω, what is the resulting resistance? ____ Ω

Checkpoint 3.5.15 We need a resulting resistance of 2.0Ω If one resistor in parallel is 6.0Ω, what should the other resistance be? ____ Ω

Checkpoint 3.5.16 If we need 5.0 Ohm resistance and one of our resistors is a 4.0 Ohm resistor, is there a second resistor to make this work? Explain.

3.5.2 Rates

There are many times when we need to calculate the rate at which something can be accomplished when more than one person/thing is working on it. This section illustrates how to obtain a resulting rate from the individual rates. This process requires algebra similar to that above.

Example 3.5.17 Joint Work: Draining Basement. A company has two pumps available for draining flooded basements. One pump can drain a basement in 4 hours, whereas the other pump can do the job in only 3 hours. How long would it take to drain the basement if both pumps are used simulataneously?

The question is how to calculate the speed of the combined pumps. Adding a pump should increase the speed. Thus we want to determine a way to add the speeds. We can start by writing down the rates to see what that suggests. The first pump operates at a rate of $\frac{1 \text{ basement}}{4 \text{ hours}}$ and the second pump operates at a rate of $\frac{1 \text{ basement}}{3 \text{ hours}}$. Because rates are ratios (fractions), and the units match, we know how to add them. The combined rate is

$$\begin{aligned}
\text{Joint Rate} &= \frac{1 \text{ basement}}{4 \text{ hours}} + \frac{1 \text{ basement}}{3 \text{ hours}} \text{ Need a common denominator} \\
&= \frac{3}{3} \cdot \frac{1 \text{ basement}}{4 \text{ hours}} + \frac{4}{4} \cdot \frac{1 \text{ basement}}{3 \text{ hours}} \text{ Scaling accomplishes this} \\
&= \frac{3 \text{ basement}}{12 \text{ hours}} + \frac{4 \text{ basement}}{12 \text{ hours}} \\
&= \frac{7 \text{ basement}}{12 \text{ hours}}.
\end{aligned}$$

We did not convert to decimal notation, because we will be setting up a proportion, so we want the rate in fraction form.

The next step is to use this rate to determine how long it takes to empty one basement. This is setting up a proportion.

$$\begin{aligned}
\frac{7 \text{ basement}}{12 \text{ hours}} &= \frac{1 \text{ basement}}{N \text{ hours}}. \\
\frac{12 \text{ hours}}{7 \text{ basement}} &= \frac{N \text{ hours}}{1 \text{ basement}}. \\
\frac{12 \text{ hours}}{7 \text{ basement}} \cdot (1 \text{ basement}) &= \frac{N \text{ hours}}{1 \text{ basement}} \cdot (1 \text{ basement}). \\
1.714285714 \text{ hours} &\approx N. \\
1.72 \text{ hours} &\approx N.
\end{aligned}$$

Given this is in hours it makes sense to round to the hundredths (a little less than a minute). It also makes sense to round up (better to expect the worst and be happy it was better). We expect the two pumps to complete the work in 1.72 hours (a little less than 1 hour and 45 minutes).

Because the denominator was 1 basement, when we cleared the denominator (cross multiplication) all we did was adjust the units (basement/basement divides out). This suggests we could have simply scaled the combined rate to obtain the final result.

$$\frac{1/7}{1/7} \cdot \frac{7 \text{ basement}}{12 \text{ hours}} = \frac{1 \text{ basement}}{1.72 \text{ hours}}.$$

Thus if both pumps are working it will take 1.72 hours drain the basement.

The increase in speed results in less time required to complete the job. This is why the faster rate (1 basement per 1.72 hours) has a smaller denominator (1.72 vs 3 or 4). ▲

Example 3.5.18 How to Use an Example: Joint Work. If one housekeeper can clean a hotel room in 11 minutes, and another can clean a room in 13 minutes. How long will it take them combined to clean 27 rooms?

Because the question asks us to determine combined speed, we recognize this as a "joint work" problem. Looking at Example 3.5.17 we see that the first step was to write down the two rates. $\frac{1 \text{ room}}{11 \text{ minutes}}$ and $\frac{1 \text{ room}}{13 \text{ minutes}}$.

After writing the rates, we realize we need to add the rates.

$$\text{Joint Rate} = \frac{1 \text{ room}}{11 \text{ minutes}} + \frac{1 \text{ room}}{13 \text{ minutes}} \text{ Need a common denominator}$$

$$= \frac{1 \text{ room}}{11 \text{ minutes}} \cdot \frac{13}{13} + \frac{1 \text{ room}}{13 \text{ minutes}} \cdot \frac{11}{11} \text{ This scales the rates}$$
$$= \frac{13 \text{ rooms}}{143 \text{ minutes}} + \frac{11 \text{ rooms}}{143 \text{ minutes}}$$
$$= \frac{24 \text{ rooms}}{143 \text{ minutes}}.$$

After calculating the rate, the next step is setting up the proportion. In this case we want to calculate the time to clean 27 rooms.

$$\frac{24 \text{ rooms}}{143 \text{ minutes}} = \frac{27 \text{ rooms}}{N \text{ minutes}}.$$
$$\frac{143 \text{ minutes}}{24\text{room}} = \frac{N \text{ minutes}}{27 \text{ rooms}}.$$
$$\frac{143 \text{ minutes}}{24 \text{ rooms}} \cdot (27 \text{ rooms}) = \frac{N \text{ minutes}}{27 \text{ rooms}} \cdot (27 \text{ rooms}).$$
$$140.875 \text{ minutes} = N.$$
$$141 \text{ minutes} \approx N.$$

Because this is about the time required to complete a task it makes sense to round up. ▲

Checkpoint 3.5.19 If one shop can do 9 float changes in two days, and a second shop can do 16 float changes in three days, how long will it take the pair of shops to do 65 float changes? ____

If they start on Monday and work only weekdays, on what day of the week will they finish?

1. Monday
2. Tuesday
3. Wednesday
4. Thursday
5. Friday

3.5.3 Exercises

1. A known volume of ideal gas is contained in a cylinder fitted with a piston and held at constant pressure. If the temperature changes from 25°C to 200.°C, by what factor does the volume change? Round your answer using the rules for working with significant figures.

For this problem you should know that when pressure is held constant, gasses follow Charles' Law, which states $\frac{V_1}{T_1 + 273} = \frac{V_2}{T_2 + 273}$, where V_1 and T_1 is the initial volume and temperature and V_2 and T_2 is the new volume and temperature.

The volume

1. decreases
2. increases

by a factor of _____

2. The ideal gas law can be expressed as $PV = nRT$ where P is pressure in atm (atmospheres), V is volume in liters, n is the number of moles (a count of molecules), R is a constant from physics, and T is the temperature (from absolute zero).

What is the pressure, in atmospheres, of 7.37 moles of a gas at a temperature of 418 K in a 227 liter container when R = 0.0821 (atm \times L)/(mol \times K)?

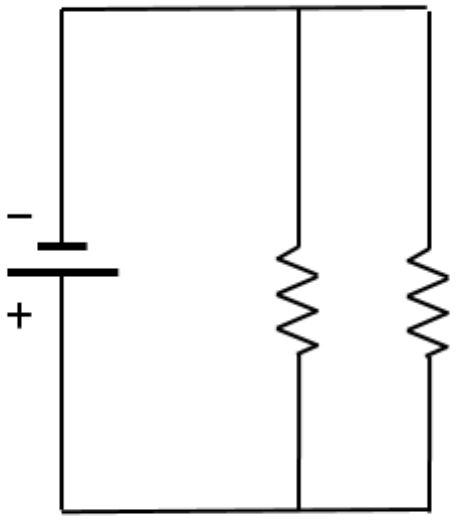
_____ atm

3. A relationship between the outer D_o and pitch D_p diameters of a gear and the number of teeth N is given by $D_p = \dfrac{D_o N}{N+2}$.

If a gear needs $N = 24$ teeth and pitch diameter $D_p = 32$ mm, what should the outer diameter be? Note that measurements are given to two significant digits and number of teeth is a count.

$D_o =$ ____

4. What is the total resistance of the circuit shown below? Use the formula $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ to answer this question, where R is the total overall resistance and R_1 and R_2 are individual resistors in parallel.

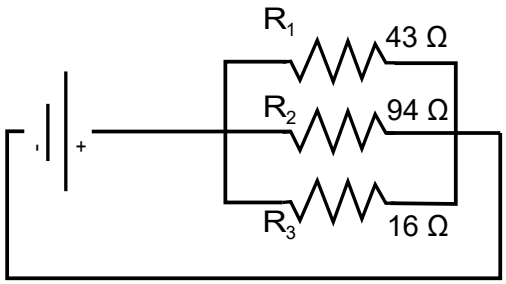


Round your answer using the rules of working with measurements.

$R_T =$ _____ unit_____

To enter the unit, in the pop-up box click on the heading "vars".

5. What is the total resistance of the circuit shown below which contains three resistors in parallel with resistance values of $R_1 = 43\ \Omega$, $R_3 = 94\ \Omega$, $R_3 = 16\ \Omega$.?

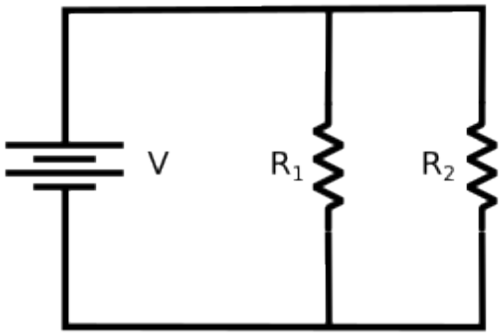


Round your answer to two decimal places, filling in with zeros as needed.

To enter the unit, click the "vars" tab and select the appropriate unit.

$R_T =$ _____ unit_____

6. Consider the following diagram of a parallel circuit.



V = 25.0 Volts
$R_1 = 250.\Omega$
$R_2 = 10\overline{0}0\Omega$
Using the information above, determine the total (equivalent) resistance, and the current.
R_{eq} = _____ Ω
I = _____

7. A contractor finds that Crew A takes 4.50 hours to construct a retaining wall and Crew B can do the same job in 6.50 hours. If Crew A and Crew B work together, how long will it take them to construct the retaining wall?

Round using the rules of significant figures.

You may find it useful to know the work formula is $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$.

Working together, the two crews will construct the wall in _____ hours.

8. Angela can shovel the snow from her driveway in 3.00 hours. When Franklin joins her, the driveway can be finished in just 54 minutes.

Working alone, Franklin can shovel the driveway in _____ minutes. Round your answer to the nearest minute.

9. You know it takes 12.5 hours to fill your pool with your hose. Last year, you used both your hose and a hose borrowed from a neighbor, and it took 8.50 hours. This year, your faucet is broken and you can only use your neighbor's hose.

Round appropriately using significant figures.

Just using the neighbor's hose, it will take _____ hours.

10. Three hoses are used to fill a swimming pool. The first hose can fill the pool in 38 minutes, the second can fill the pool in 42 minutes, and the third can fill the pool in 51 minutes. All numbers have two significant digits.

Using all three hoses, the pool will be filled in _____ minutes.

11. A gear with 35 teeth directly drives another gear with 15 teeth. If the larger gear spins at 470 rpm, at what speed will the smaller gear spin? _____

12. A gear with 40 teeth is connected via a belt to another gear with 20 teeth. If the larger gear spins at 440 rpm, at what speed will the smaller gear spin? _____

13. A gear with 40 teeth directly drives a smaller gear. If the larger gear spins at 410 rpm, and the smaller gear must spins at 1093.3333333333, how many teeth should the smaller gear have? _____

14. On a bicycle the pedals move a chain ring with 50 teeth. The chain moves a gear on the cassette which has 23 teeth. The chain also goes over a chain pulley with 10 teeth.

If the cyclist is pedalling at 89 rpm, at what rpm are the other gears spinning? Round to the nearest integer

gear on cassette _____

pulley _____

15. Re-write the following as a polynomial equal to zero.

$$\frac{(1+x)(1+x)}{(1-x)(1-x)} = 1.40$$

$0 =$ _____

16. Re-write the following as a polynomial equal to zero.

$$\frac{(2x)^2}{(1.600-x)(1.500-x)} = 115$$

$0 =$ _____

17. Solve: $I = \dfrac{V}{R+r}$ for R.

$R =$ _____

3.6 Linear Systems

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Solve a system of linear equations (skill)

In Example 2.2.1 we solved a problem with two, different mixtures. In that case we knew how much of each we were adding. In other cases we know what result we want and need to figure out how much of each substance. This section presents a method for answering questions like this involving two or more constraints (equations) at the same time.

3.6.1 Motivation

This section provides a first examples of a problem involving two, linear equations. It illustrates how we would recognize this type of problem and how to write it.

In previous dilution problems, we diluted a mixture using just diluent (water in that case). In other situations we will have two mixtures with different percents that we will combine to obtain a new mixture.

Example 3.6.1 Combining Mixtures. Suppose we have 16 oz of 91% isopropyl alcohol and 12 oz of 75% isopropyl alcohol. How much of each do we need to mix to produce 10.0 oz of 85% alcohol?

The question asks us to determine an amount for each of the two mixtures. We can use a common technique in mathematics which is to start by writing the answer. We will declare that we will use A oz of 91% alcohol and B oz of 75% alcohol. Next we can look at the constraints and use them to write equations using these variables.

The first constraint is that we end up with 10 oz of solution. Thus when we add the two mixtures we end up with 10 oz, which is expressed by

$$A + B = 10.0.$$

The second contraint is the percent alcohol. Because our variables are in terms of amount of solution, we need to express the percent alcohol constraint in terms of ounces (not percents). We can obtain amount from percent using the definition of percent. Because the resulting solution will be 85% alcohol there will be

$$(0.85)10.0 = 8.5 \text{ oz.}$$

This will be the result of adding the amounts from each solution (just as in previous mixture problems). Because A oz of the first solution will be added and it is 91% alcohol, it will contribute $(0.91)A$ oz of alcohol. Similarly the second solution will contribute $(0.75)B$ oz of alcohol. Combined we will obtain

$$(0.91)A + (0.75)B = 8.5 \text{ oz.}$$

Now we just need a way to solve this pair of equations. ▲

In the next example two business considerations lead to a linear system.

Example 3.6.2 Based on past sales, a business determines that snack fans prefer two item trail mix consisting of cashews and dried blueberries in 6 oz portions. They are willing to pay \$9/lb for it. If the cashews cost \$8.50/lb and the dried blueberries cost \$9.75/lb, how many oz of each should be in one 6 oz portion?

We are asked for amounts for both ingredients. Let's call the number of ounces c and b. Before we set up equations we should convert the prices to dollars per ounce.

$$\frac{\$8.50}{\text{lb}} \cdot \frac{1 \text{ lb}}{16 \text{ oz}} = \$0.53125/\text{oz}.$$

$$\frac{\$9.75}{\text{lb}} \cdot \frac{1 \text{ lb}}{16 \text{ oz}} = \$0.609375/\text{oz}.$$

$$\frac{\$9.00}{\text{lb}} \cdot \frac{1 \text{ lb}}{16 \text{ oz}} = \$0.5625/\text{oz}.$$

The first constraint is the total amount. This is

$$c + b = 6 \text{ oz}.$$

The second constraint is the cost. Total cost (\$9.00) is the sum of the cost of the two ingredients, so the equation is

$$\frac{\$0.53125}{\text{oz}}(c \text{ oz}) + \frac{\$0.609375}{\text{oz}}(b \text{ oz}) = \$0.5625.$$

▲

The next example illustrates what to do when one or more of the constraints do not use all the variables. Note, the question can be answered without a linear system (see Section 2.2); we use it here as a familiar example.

Example 3.6.3 Suppose we have 16 oz of 91% isopropyl alcohol and 16 oz of water. How much of each do we need to mix to produce 10.0 oz of 85% alcohol?

This can be set up the same way as Example 3.6.1. The one change is that the second mixture is 0% isopropyl alcohol. This changes the equations to

$$\begin{aligned} A + B &= 10.0 \\ (0.91)A + (0.00)B &= 8.5. \end{aligned}$$

The second equation can be re-written as

$$0.91A = 8.5.$$

We can now solve as we did before or using a method shown below. ▲

The following example illustrated both writing a system of linear equations and also how to use multiple examples to answer a question.

Example 3.6.4 A plane has to fly 106 nm in a straight line between Merrill Field and Homer airports. If the flight from Merrill to Homer took 53 minutes, and the flight from Homer to Merrill took 45 minutes, how fast was the plane flying through the air and how fast was the head/tail wind? We will suppose the wind speed was the same the whole time.

At first it is not clear what type of calculation we will need. We can follow the advice in Example 3.6.1 to simply write down the answer. Because we are asked for plane and wind speeds, these are the variables. Call them p and w.

We also have data on two flights. These include times and distance but not speed (what we are asked to find). So we must ask ourselves how we can convert time and distance into speeds. We did this in Example 2.4.4 and Example 2.4.6.

For the trip to Homer we would like to setup

$$\frac{S_H \text{ nm}}{\text{hr}} = \frac{106 \text{ nm}}{53 \text{ min}},$$

however this has hours on the left and minutes on the right. So we must convert the time in minutes to time in hours (unit conversion is in Section 1.1). Because there are 60 minutes per hour, 53 minutes is 53/60 hours.

$$\begin{aligned}\frac{S_H \text{ nm}}{\text{hr}} &= \frac{106 \text{ nm}}{53/60 \text{ hr}}. \\ &= 120\frac{\text{nm}}{\text{hr}}.\end{aligned}$$

For the return trip (Homer to Merrill) we perform the same calculation.

$$\frac{S_M \text{ nm}}{\text{hr}} = \frac{106 \text{ nm}}{45/60} \approx 141\frac{\text{nm}}{\text{hr}}.$$

Why do we have two speeds for the same trip? The wind slows down the first flight and speeds up the second. The speeds calculated above are across the ground (nm across the ground per hour). When travelling with wind at our backs, our speed is increased by the wind speed. When travelling with a head wind, our speed is decreased by the wind speed. For example if a plane is flying at 60 nm/hr and has a 7 nm/hr head wind it will be travelling only $60 - 7 = 53$ nm/hr over the ground. This idea finally enables us to setup equations for the constraints.

$$\begin{aligned}p - w &= 120. \\ p + w &= 141.\end{aligned}$$

Later in this section we will learn a technique to solve these equations. However, this is a special case. The speed p must be between 120 and 141. Because the same amount (w) is subtracted and added, p is half-way between or

$$p = \frac{120 + 141}{2} \approx 130.5 \approx 130.$$

Rounding to 130 is using significant digits rounding (times given had only 2 significant digits). ▲

3.6.2 Crossing Lines

This section connects solving a system of two linear equations to their graphs. Graphs will help us understand why (and when) there should be a solution. The next section provides the method for solving.

Our goal here is to consider what causes lines to cross. We will do this by looking at a pair of lines and seeing where they cross.

Recall that a line is a relation (set of points) such that the change between any two, equally spaced points is the same. Often you have heard this described as rise over run or slope. Slope is a geometric interpretation referring to how **steep** the line is.

Checkpoint 3.6.5 In Figure 3.6.6 there are two lines. One goes through point $A = (1, 3)$. It rises at a slope of 2/1 (two up for each one over). The other line goes through the point $B = (1, 2)$ which is below A.

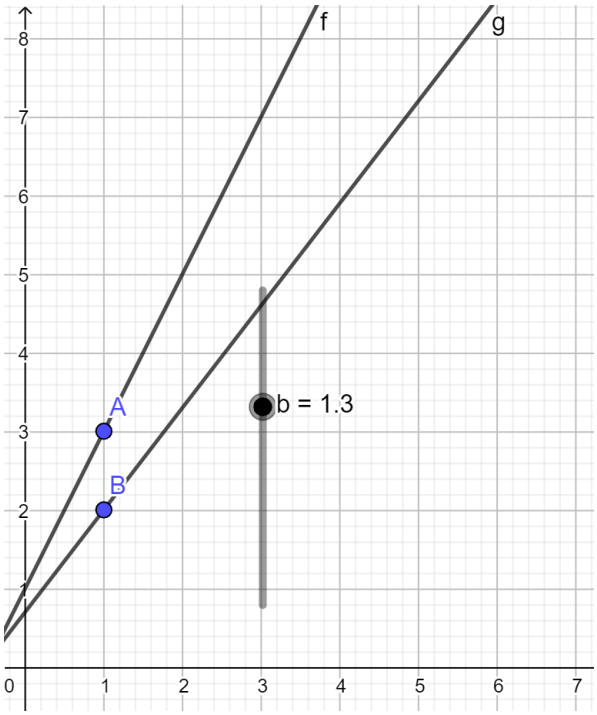
(a) Use the slider to set the slope of the second line to 3. Does the line cross the one through A? Where (left or right of point B)?

(b) Use the slider to set the slope to something bigger than 3. Does the line cross the first one? Where (left or right of point B)?

(c) Use the slider to set the slope to 1. Does the line cross the first one? Where (left or right of point B)?

(d) In general if either slope is steeper than the other slope will the two lines cross?

(e) Can you select a slope for the second line so that these two lines do not cross?



Standalone
In Context
Embed

Figure 3.6.6 Crossing Lines

This next example applies the idea of a line starting lower but rising faster to answer a financial question.

Example 3.6.7 Vasya's initial pay was \$62,347.23. She received \$5,000 raises each year. Pyotr's initial pay was \$67,242.33. He receives \$3,500 raises each year. If they were both hired in 2012 in what year does Vasya first have a higher salary?

We could make a table.

Year	Vasya	Pyotr
2012	\$62,347.23	\$67,242.33
2013	\$67,347.23	\$70,742.33
2014	\$72,347.23	\$74,242.33
2015	\$77,347.23	\$77,742.33
2016	\$82,347.23	\$81,242.33

We see that is in 2016 that Vasya is first paid more.

Alternatively we could note that Vasya's raises are \$1,500 more each year than Pyotr's raises. This means she closes the gap by \$1,500 each year. The difference in their initial salaries is $67242.33 - 62347.23 = 4895.10$. Because she gains by 1500 each year it will take $4895.10/1500 = 3.2634$ years. Because they receive raises once a year this result must be rounded up to 4 years. Thus she is first paid more in 2016. ▲

3.6.3 Solving Linear Systems

This section presents two ways of solving linear systems of this type. The second method is very important for larger systems.

Example 3.6.8 We will solve the system from Example 3.6.1. The two equations are

$$A + B = 10.0.$$
$$(0.91)A + (0.75)B = 8.5.$$

Notice we can solve the first equation for B. If we do then we can substitute it into the second equation. The result will be a single equation with only one variable. We already know how to solve equations like that.

$$\begin{aligned} A + B &= 10.0. \\ B &= 10.0 - A. \end{aligned}$$

$$\begin{aligned} (0.91)A + (0.75)B &= 8.5. && \\ (0.91)A + (0.75)(10.0 - A) &= 8.5. && \text{Substitute B from above} \\ (0.91)A + 7.5 - (0.75)A &= 8.5. && \text{Distribute} \\ (0.91)A - (0.75)A &= 8.5 - 7.5. && \\ (0.91 - 0.75)A &= 1.0. && \\ (0.16)A &= 1.0. && \\ A &= \frac{1.0}{0.16} && \\ &= 6.25. && \end{aligned}$$

Now that we know that $A = 6.25$ we can substitute that into $A + B = 10.0$. This gives us

$$\begin{aligned} A + B &= 10.0. \\ 6.25 + B &= 10.0. \\ B &= 3.75. \end{aligned}$$

We can check that this works in the other equation (about percent alcohol).

$$\begin{aligned} (0.91)A + (0.75)B &= \\ (0.91)(6.25) + (0.75)(3.75) &= \\ 5.6875 + 2.8125 &= 8.5. \end{aligned}$$

▲

If we had 7 variables instead of two, substituting would take a while. Instead we can use the following method which is more like solving as we know it, that is isolating a variable. This method is called **elimination**.

Example 3.6.9 We will solve the system

$$\begin{aligned} A + B &= 10.0. \\ (0.91)A + (0.75)B &= 8.5. \end{aligned}$$

In the second line below notice how we modify the first equation to partially match the second one.

$$\begin{aligned} A + B &= 10.0. && \text{Original equation 1} \\ -(0.91)(A + B) &= -(0.91)10.0. && \text{Scale to match 2nd equation} \\ -(0.91)A - (0.91)B &= -9.1. && \text{Distribute} \end{aligned}$$

Now add the two equations.

$$\begin{aligned} -(0.91)A - (0.91)B &= -9.1. && \text{Scaled equation 1} \\ (0.91)A + (0.75)B &= 8.5. && \text{Original equation 2} \\ -(0.16)B &= -0.6. && \text{Result of adding previous 2 lines} \\ B &= \frac{-0.6}{-0.16} && \end{aligned}$$

$$= 3.75.$$

In the fifth line we added the two equations. Because they had opposite coefficients for A, that variable was eliminated, leaving us with just B. This can always be done with systems of linear equations.

We finish solving this system the same way as the previous example, by substituting the value of B back into the first equation.

$$\begin{aligned} A + B &= 10.0. \\ A + 3.75 &= 10.0. \\ A &= 6.25. \end{aligned}$$

▲

Checkpoint 3.6.10 Solve this system of equations using substitution.

$-9x + 9y = 45$

$6x - y = -25$

_____

Checkpoint 3.6.11 Solve this system of equations using elimination.

$-x + 8y = -21$

$8x + 6y = 28$

_____

3.6.4 Other Cases

In Figure 3.6.6 we discovered a slope that resulted in no intersection. If we were solving a pair of linear equations that represented lines like this, our calculations would not produce a solution. These are known as **inconsistent** systems. This section provides two examples of such systems and demonstrates how we identify them. For this book, identifying these cases and correctly describing them is all you are expected to do.

Example 3.6.12 Inconsistent Linear System. Find all solutions to the system

$$\begin{aligned} 2x + 3y &= 5. \\ 4x + 6y &= 7. \end{aligned}$$

We will use elimination. If we multiply -2 by the first equation we will obtain -4 (opposite of x in the second equation).

$$\begin{aligned} 2x + 3y &= 5. \\ -2(2x + 3y) &= -2(5). \\ -4x - 6y &= -10. \\ 4x + 6y &= 7. \\ 0 &= -3. \end{aligned}$$

Our work is correct, but the conclusion is clearly false. You can think of this as saying, for a solution to exist 0 must equal -3. Because this is a contradiction, we call the system **inconsistent**. This means there are no solutions. ▲

There is a third case.

Example 3.6.13 Dependent System. Find all solutions to the system

$$\begin{aligned} 2x + 3y &= 5. \\ 4x + 6y &= 10. \end{aligned}$$

We will use elimination. If we multiply -2 by the first equation we will obtain -4 (opposite of x in the

second equation).

$$\begin{aligned} 2x+3y &= 5. \\ -2(2x+3y) &= -2(5). \\ -4x-6y &= -10. \\ 4x+6y &= 10. \\ 0 &= 0. \end{aligned}$$

This time we have a true, but rather uninformative statement. We notice that after scaling (multiplying by -2) the two equations were identical. Essentially we had only one equation. Because one can be obtained from the other we call them **dependent**. ▲

Checkpoint 3.6.14 Determine what type of linear system this is.
$5x-6y=-89$
$15x-18y=-267$

1. Consistent
2. Inconsistent
3. Dependent

3.6.5 Exercises

Solve Systems of Equations. Solve each of the following linear systems.

1. Solve this system of equations
$$\begin{cases} 2x-y &= 8 \\ 2x+3y &= -8 \end{cases}$$
_____

2. Solve this system of equations
$$\begin{cases} -4x+4y &= -32 \\ x-4y &= 23 \end{cases}$$
_____

3. Solve the system using addition/elimination method. If there is exactly one solution, write as an ordered pair. If not, choose one of the other options.
$$\begin{cases} x+4y &= -17 \\ -x+y &= 2 \end{cases}$$

- One solution: _____
- No solution
- Infinite number of solutions

4. Solve the given linear system of equations:
$$\begin{cases} 2x-8y &= 5 \\ -3x+12y &= 15 \end{cases}$$

- One or more solutions: _____
- No solution
- Infinite number of solutions

5. Solve the following system.

$$\begin{aligned} y &+ 5z = -1 \\ 2y &- 2z = -3 \end{aligned}$$

Solution: ___

Enter your answers as integers or reduced fractions.

6. Solve the system using addition/elimination method. If there is exactly one solution, write as an ordered pair. If not, choose one of the other options.

$$\begin{aligned} -2a &+ 6b = 6 \\ 6a &- 18b = -18 \end{aligned}$$

- One solution: _____
- No solution
- Infinite number of solutions

7. Solve the system of equation.

$$\begin{aligned} x + y &= -8 \\ -2y - 2x &= 16 \end{aligned}$$

- One solution_____
- No solution
- Infinitely many points

Applications. Identify the linear system and solve it to answer these questions.

8. We have a jar of coins, all dimes and pennies. All together, we have 225 coins, and the total value of all coins in the jar is \$6.84. How many dimes are there in the jar?

____ dimes and ____ pennies

9. A manufacturer has just announced their new snowboard. The accounting department has determined that the cost to manufacturer the snowboard is $c = 33.75n + 40256$, where n is the number of units manufactured and c is the resulting cost. The revenue equation is $c = 96.65n$. What is the break even point? _____

10. A store owner wants to mix chocolate and nuts to make a new candy. How many pounds of chocolate costing \$17.70 per pound should be mixed with 14 pounds of nuts that cost \$2.60 per pound to create a mixture worth \$9.57 per pound?

The owner needs to mix _____ pounds of chocolate.
(round to the nearest whole pound)

11. A coffee distributor plans to mix some Mexican Shade Grown coffee that sells for \$9.20 per pound with some Terraza coffee that sells for \$13.00 per pound to create 10 pounds of a new coffee blend that will sell for \$12.24 per pound.

How many pounds of each kind of coffee should they mix? Round to the nearest pound.
____ pounds of Mexican Shade Grown coffee.
____ pounds of Terraza coffee.

12.



*2540*The return trip takes *3* hours flying against the wind.
What is the speed of the airplane in still air and how fast is the wind blowing?
Answer:
The speed of the airplane in still air is _____ miles per hour.
The wind speed is _____ miles per hour.
Round your values to the nearest whole number.

13.



A certain bread recipe asks you to combine flour and yeast with *4* cups of warm *115* °F water. If the water is hotter or colder than that, then the bread won't rise.

All you have available are water from your instant hot water dispenser that is *200* °F and refrigerated water that is *40* °F.

How much water from your instant hot water dispenser and refrigerated water should you mix together to get *4* cups of *115* °F water?

Answer:
____ cups of water from your instant hot water dispenser.
____ cups of refrigerated water.
Round your answers to 2 decimal places.

14. A 5.00% solution of pesticide and a 7.00% solution of pesticide must be combined to produce 147 mL of a 5.94% solution. How much of each type should be mixed? Round using the rules of working with significant figures.
_____ mL of 5 %
_____ mL of 7 %

15. At a farmers' market, Frederick buys 4 pounds of apples and 12 pounds of cherries for $33.20. At the same farmers' market, Wilhelmina buys 12 pounds of apples and 9 pounds of cherries for $33.72. Determine the price per pound of apples and cherries at the farmers' market.

Apples cost $_____ per pound.
Cherries cost $_____ per pound.

3.7 Project: Radiation Dosage

Project 3.7.1 Calculating Effects of Radiation. The purpose of this project is to build a mathematical model for a situation. We will focus on the structure of the equations, and what they tell us about the mathematical relationships of the data. The emphasis is not on actual numbers.

For an unrelated example of a model, consider dropping a ball off a cliff. Ignoring air resistance, the ball's position can be modeled by the equation $h = kt^2$, where t is the time and h is the height of the ball. There are some other numbers that go in there, but what is changing is the time and position (height). That equation framework is the mathematical model.

You may find this to be a challenging project. Do the best you can and use your own common sense. Math should make sense! After you finish this, please read back over your work and make sure your answers are logically consistent.

Remember that you can ask questions and meet up with a tutor, but you should *not* be looking up answers or just writing down what someone else says. Do not let someone else copy your answers. That is academic dishonesty and you should not allow it. Your work should be your own.

For this project, imagine that you are working with radioactive material. Since we do not want you harmed by radiation, we should understand how time and distance impacts the radioactive dose. (You should also be behind material that shields you from the radiation, but that math is more complicated, so we'll focus on time and distance.)

(a) As your time near a radioactive source increases, does the radiation level in your body increase, decrease, or stay the same?

(b) Do you think the relationship between time and dosage is linear or not linear?

(c) Let E stand for radiation exposure, t stand for length of time of exposure, and k be the constant of variation. Write an equation representing the relationship between E, t, and k.

(d) Examine the above equation you just wrote.

(a) Is it linear or non-linear? This should match your answer to Task 3.7.1.b.

(b) What in the equation indicates it is linear or non-linear? That is, how did you know the answer to part Item 1?

(e) We have determined the relation between time and radiation exposure. Next we determine the relation between distance to radiation source and radiation exposure. These can be different.

Radiation radiates outwards from a source evenly in all directions (like light radiates out evenly from a lightbulb) unless it is obstructed by something (like a lead shield). Imagine a radiation source floating in the center of a sphere. All parts of the sphere would be getting hit with an equal amount of radiation. We are going to figure out the radiation for a given patch of area on this sphere.

It will be helpful to know the following formula: $A = 4\pi r^2$, where A is the surface area of the sphere and r is the radius of the sphere.

If a sphere had a radius of 2 m, what is the surface area of the sphere? Remember to include units. Leave your answer in terms of π (meaning it should look like _____π m^2).

(f) A very rough approximation of the surface area of the front of a person is 1 m^2. We will consider what percent of the surface area of a sphere this is.

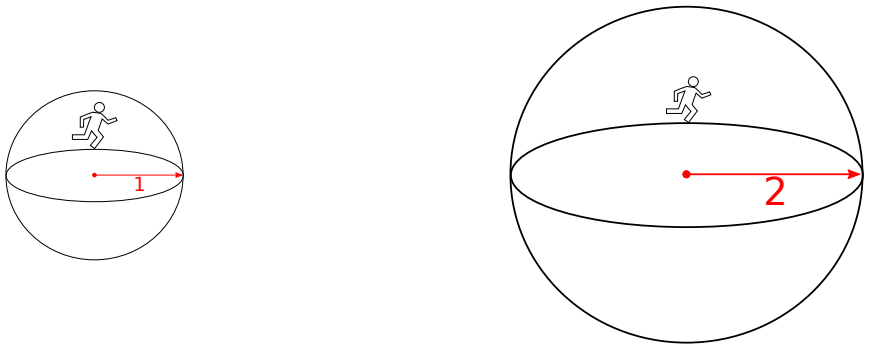


Figure 3.7.2 Surface Area Ratio

(i) What percent of the surface area of a sphere of radius 1 is $1\,\text{m}^2$?

(ii) What percent of the surface area of a sphere of radius 2 is $1\,\text{m}^2$?

(iii) Why does the change in percent make sense? See Figure 3.7.2.

(iv) In general as the radius increases what will the percent of the surface that is $1\,\text{m}^2$ do? Increase/remain the same/decrease?

(g) Note that the amount of radiation (energy) remains the same regardless of the radius of the sphere. That is a sphere with radius 1 and surface area 4π has the same energy as a sphere with radius 2 and surface area 16π.

(i) If the radiation is being emitted at an intensity of 5 Sieverts per hour ($\frac{\text{Sv}}{\text{h}}$), what amount of radiation will be hitting our $1\,\text{m}^2$ person who is at a distance of 1 m from the source?

(ii) If the radiation is being emitted at an intensity of 5 Sieverts per hour ($\frac{\text{Sv}}{\text{h}}$), what amount of radiation will be hitting our $1\,\text{m}^2$ person who is at a distance of 2 m from the source?

(iii) In general as the distance (radius) increases what happens to the amount of radiation absorbed by the person do? Increase/remain the same/decrease?

(h) If the radiation is being emitted at an intensity of x Sieverts per hour ($\frac{\text{Sv}}{\text{h}}$), what amount of radiation will be hitting our human-sized cutout on the surface of the sphere? Your answer should be in terms of x.

(i) Complete the following table. Notice that you already found the values for the first row.

Table 3.7.3 Ratio of Surface Areas

Radius of circle	Ratio of $1\,\text{m}^2$ to surface area
2 m	
3 m	
4 m	
5 m	
r m	

(j) Graph the points in Table 3.7.3.

Hint. Your horizontal axis (radius) should be from 0 to 5. Because the output numbers are small, we need a scale that matches. Make the units on the vertical scale 0.002.

(k) Does the data represented in the table above represent a linear relationship or a non-linear relationship? Give a reason to justify your answer.

(l) Is the relationship between distance and the potential amount of radiation hitting the person better modeled by direct variation or inverse variation?

(m) Consider the numbers 16, 36, 64, 100. These numbers are significant in mathematics. What is the pattern or significance of these numbers? (Note: you only see these numbers if you completed the table in terms of π (do not multiply and round). Go back and fix your table if you do not see these numbers.)

(n) As the radius of the sphere increases, does the level of radiation hitting our person-sized cutout increase or decrease? Does this increase or decrease change linearly (at a constant rate) or non-linearly (at a changing rate)? Circle the appropriate answer.

As the radius of the sphere increases, the radiation intensity is *increasing / decreasing* (circle one) in a *linear / non-linear* (circle one) fashion.

(o) The sphere with a floating radiation source is a good model for us to use when thinking through how distance impacts radiation levels because we can disregard complicating factors like the walls, floor, and ceiling of the room as long as a person is still directly exposed to the radiation source. Still, the relationship you found between radius and radiation holds true. With that in mind, answer the following questions.

Let I stand for radiation experienced by the person, r stand for distance, and k be the constant of variation. Write an equation of variation representing the relationship between I, r, and k.

Hint. Look back at your recent answers and the table you built. Is the formula you wrote logically consistent with these answers? That is, if you plugged in the r with some constant value k, would you get the right answer for I?

(p) Let R stand for radiation exposure, t stand for length of time of exposure, r stand for distance, and k be the constant of variation (this k may be different from your k in Task 3.7.1.c or Task 3.7.1.n). Write an equation of joint variation representing the relationship between R, r, t, and k.

Hint. Is your answer consistent with your answers to #3 and #13?

(q) Use the equation you just built. If you are 5 meters away for 3 hours, how long would you stay at 2 meters away to receive the same radiation dose?

(r) At the end of it all, if you find yourself next to a radioactive source, what do you do? Full points will be awarded for all reasonable answers that address both time and distance.

3.8 Project: Biking in Kansas and Alaska

Project 3.8.1 Project: Biking in Kansas and Alaska. In this project, we are going to think about what makes a relationship linear or not linear. Each question is worth two points.

(a) This is a graph of a linear relationship. Looking at it, what about it tells you that it is linear?

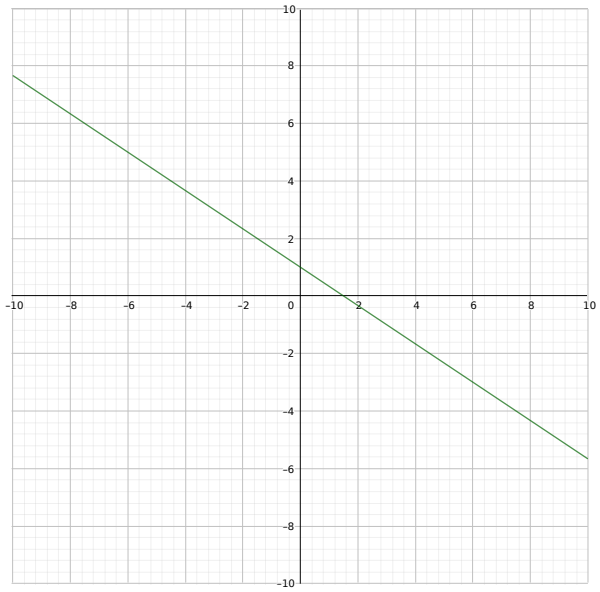


Figure 3.8.2 Graph of Line

(b) Here is a table of some of the points represented on the above graph. This data also represents a linear relationship. Without graphing, how can you tell that this relationship is linear?

Table 3.8.3 Table of Points

x	y
-6	5
-3	3
0	1
3	-1
6	-3

(c) Friends Jacob and Mike like to bike. For a math conference, the two traveled to Kansas and decided to go on a bike ride one evening. Mike enjoys tracking his data and so took note of his distance traveled at regular intervals. Here is a table of Mike's time and mileage:

Table 3.8.4 Time and Distance

Time biked (in minutes)	Distance traveled (in miles)
10	2
20	4
30	6
40	8
50	10

Does this table represent a linear relationship? Give some supporting computations OR write a sentence to support your answer.

(d) Jacob is more absent minded in tracking his mileage over time, and so took note of his distance traveled sporadically. Here is a table of Jacob's time and mileage:

Table 3.8.5 Time and Distance

Time biked (in minutes)	Distance traveled (in miles)
7	1.4
12	2.4
20	4
35	7
54	10.8

Does this table represent a linear relationship? Give some supporting computations OR write a sentence to support your answer.

(e) After returning home to Alaska, the friends decide to go on another ride. This bike ride was on a trail in the foothills of the Chugach Mountains. Again, Mike took note of his distance traveled at regular intervals Here is a table of Mike's time and mileage:

Table 3.8.6 Time and Distance

Time biked (in minutes)	Distance traveled (in miles)
10	2.3
20	4
30	5.2
40	6
50	8

Does this table represent a linear relationship? Give some supporting computations OR write a sentence to support your answer.

(f) Again, Jacob is absent minded in tracking his mileage over time, and so took note of his distance traveled sporadically. Here is a table of Jacob's time and mileage:

Table 3.8.7 Time and Distance

Time biked (in minutes)	Distance traveled (in miles)
11	2.5
20	4
25	4.8
43	7
65	10.5

Does this table represent a linear relationship? Give some supporting computations OR write a sentence to support your answer.

(g) Slope is $\frac{\text{change in y values}}{\text{change in x values}}$. If the first columns of the four tables above represent x values and the second columns represent y values, determine the unit of the slope. Your answer should be a unit, like $\frac{\text{ft}}{\text{s}}$ or in^2, *not a number*.

(h) (1 point extra credit): Consider your answer to the previous question. What does this unit represent? Your answer can be one word.

3.9 Project: Constraints on Dilution Problems

Project 3.9.1 Project: Constraints on Dilution. Curiosity is an important mathematical virtue. We have seen limitations on the results we can obtain when diluting a mixture. This project guides us through conclusions on those constraints and asks at what rate it grows.

In Section 2.2 we learned to calculate percents for mixtures and how to dilute a mixture to a specified

percent.

By adding water we can of course not increase the percent alcohol, so if we start with 91% alcohol 91% is the highest we can achieve. On the other side we can reduce the percent alcohol to as little as we want (not quite to 0%) if we dilute it enough. This dilution requires not restricting the final volume. This pair of restrictions should make us wonder about a relationship between the desired volume and the minimum/ maximum amount of alchol.

For all of these questions start with 16 oz of 91.0% alcohol solution.

(a) This first question is the same as Example 2.2.6. Use it to review the basic dilution calculation.

Suppose you have 16 oz of 91.0% alcohol solution. How much water must we add to obtain at least 20.0 oz of 70.0% alcohol solution?

How many ounces is the resulting solution?

(b) Next we will illustrate that for a percent alcohol such as 70%, there is a maximum volume we can achieve. Specifically this is the number we already calculated.

(i) If we dilute to a 70.0% alcohol solution what is the resulting amount of solution? You calculate this above.

Add one ounce of water. Calculate the resulting percent alcohol.

(ii) Did the percent alcohol decrease, stay the same, or increase?

(iii) As a result can we produce more 70.0% alcohol solution starting with 16.0 oz of 91.0% alcohol?

Note, if we added less water we would have less solution, so that is a decrease (not the maximum).

(c) Next, we ask at what rate does the maximum percent alcohol increase or decrease as we increase the desired amount of solution. To figure this out we will calculate the percent for multiple amounts and analyze the data as we did in Section 3.3 and Section 3.4

(i) How much water must be added for 20 oz of solution?
What is the resulting percent alcohol?

(ii) How much water must be added for 22 oz of solution?
What is the resulting percent alcohol?

(iii) How much water must be added for 24 oz of solution?
What is the resulting percent alcohol?

(iv) Does the maximum percent alcohol increase or decrease with the increase in the number of ounces?

(v) Does it grow linearly, quadratically, exponentially, or otherwise?

(d) We can ask this question in reverse as well. If we want a percent alcohol, what is the resulting maximum amount of resulting solution? Then we measure the growth of the amount.

Once again we start with 16.0 oz of 91.0% alcohol solution.

(i) If we want exactly 80.0% alcohol, what is the resulting volume of solution?

(ii) If we want exactly 70.0% alcohol, what is the resulting volume of solution?

(iii) If we want exactly 60.0% alcohol, what is the resulting volume of solution?

(iv) Does the volume of solution increase or decrease as the percent concentration decreases?

(v) Does the volume of solution grow/shrink linearly, quadratically, exponentially, or otherwise?

Note if in some model a variable a varies directly with respect to a variable b, then b must vary directly with respect to a. It cannot be direct in one direction and inverse in the other.

Chapter 4

Geometry

4.1 Geometric Reasoning 2D

This section addresses the following topics.

- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Identify shapes and apply their properties (skill)

This section presents geometric properties, illustrates identifying shapes in applications, and illustrates breaking down complex shapes into simple ones.

4.1.1 Formulae

This section defines the two properties of interest and provides the formulae for some common shapes. Memorizing all of the formulae is not likely useful: in a job you will be able to look them up. However, anything you use a lot (e.g., triangles) is worth memorizing.

Two of the properties of shapes we will consider are **perimeter** and **area**. The **perimeter** of a shape is a measure of the size of its border (edges). The **area** of a shape is a measure of what it takes to fill the shape. In the figure below showing the outline of Kapi'olani Regional Park the perimeter is the distance someone would travel walking around the edge of the park. The area determines how much grass seed it would take to spread over the park.

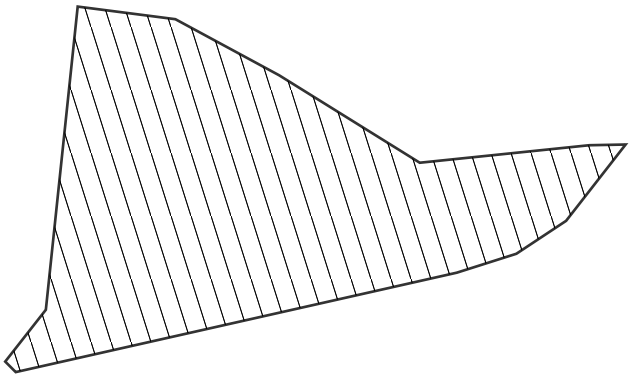


Figure 4.1.1 Complex Shape with Perimeter and Area

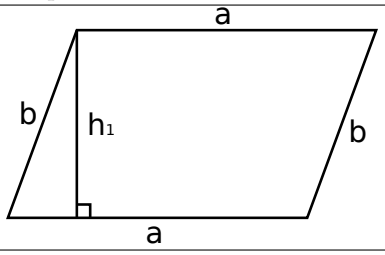
Rounding. How we round on geometric problems will depend as always on the application. For example a carpenter will round to units that can be measured with a tape measure like 1/16 of an inch. Contextless examples in this section will be rounded using significant digits. When using significant digits treat all numbers in the formulae as exact numbers.

Definition 4.1.2 Parallelogram. A **parallelogram** is a four sided shape for which opposing pairs of sides are parallel. ◆

Parallelograms include **rectangles**, which are parallelograms with four right angles, and **rhombi**, which are parallelograms with four equal length sides. A square is a rectangle and a rhombus.

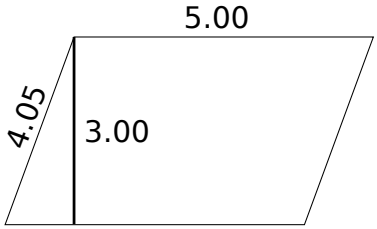
For the formulas we need a segment that is perpendicular to one pair of parallel sides. This is called the **height**. One of these parallel sides is called the **base**.

Table 4.1.3 Parallelograms

Shape	Perimeter	Area
	$2(a+b)$	$h_1 a$

The perimeter formula is the sum of the four sides. Because the sides come in two pairs which each have the same length we end up with $a+b+a+b = a+a+b+b = 2a+2b = 2(a+b)$.

Example 4.1.4 What are the perimeter and area of this parallelogram?



The perimeter is the sum of the side lengths which in this case is

$$2(4.05+5.00) = 18.1.$$

The area of a parallelogram, given in Table 4.1.3 is $h_1 a$. For this parallelogram that is

$$\text{Area} = 3.00 \cdot 5.00 = 15.0.$$

▲

Example 4.1.5 What are the perimeter and area of this parallelogram?



The perimeter is the sum of the side lengths which in this case is

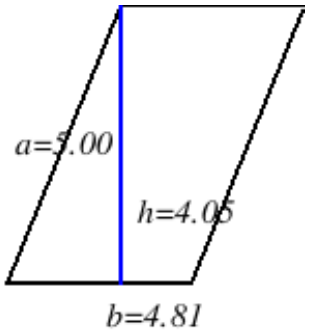
$$2(18.1 + 28.2) = 92.8.$$

The area of a parallelogram, given in Table 4.1.3 is $h_1 a$. Because this is a rectangle a side length can be used as the height.

$$\text{Area} = 18.1 \cdot 28.2 = 510.42 \approx 510.$$

▲

Checkpoint 4.1.6

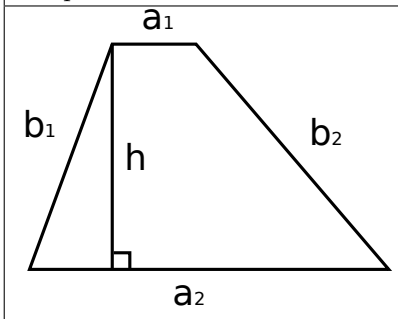


Find the perimeter and area of the parallelogram above. Side lengths are measurements.
perimeter = _____
area = _____

Definition 4.1.7 Trapezoid. A **trapezoid** is a four sided shape for which one pair of opposing sides are parallel. ◆

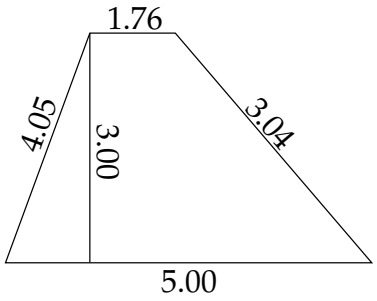
Like a parallelogram a trapezoid has a **height**. This is a segment perpendicular to the parallel sides.

Table 4.1.8 Trapezoid

Shape	Perimeter	Area
	$a_1 + b_1 + a_2 + b_2$	$\frac{h}{2}(a_1 + a_2)$

Because none of the side lengths need be the same, the perimeter "formula" is just the sum of the four sides. The height must be between the two, parallel sides. Any line between the other two sides is not perpendicular to both and the formula does not work.

Example 4.1.9 What are the perimeter and area of this trapezoid?



The perimeter of this trapezoid is the sum of the four side lengths

$$4.05 + 1.76 + 3.04 + 5.00 = 13.85.$$

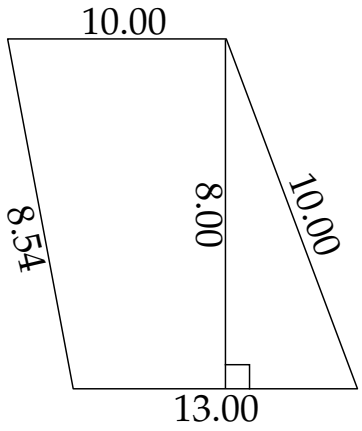
The area of a trapezoid, given in Table 4.1.8 is $\frac{h}{2}(a+b)$. For this trapezoid that is

$$\text{Area} = \frac{3.00}{2}(1.76 + 5.00) = 10.14 \approx 10.1.$$

▲

Note the following is also a trapezoid and the formulae still apply.

Example 4.1.10 What are the perimeter and area of this trapezoid?



The perimeter of this trapezoid is the sum of the four side lengths

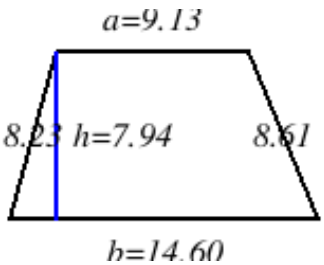
$$10.00 + 10.00 + 8.54 + 13.00 = 41.54.$$

The area of a trapezoid, given in Table 4.1.8 is $\frac{h}{2}(a+b)$. For this trapezoid that is

$$\text{Area} = \frac{8.00}{2}(10.00 + 13.00) = 92.0.$$

▲

Checkpoint 4.1.11



Find the perimeter and area of the trapezoid above. Side lengths are measurements.

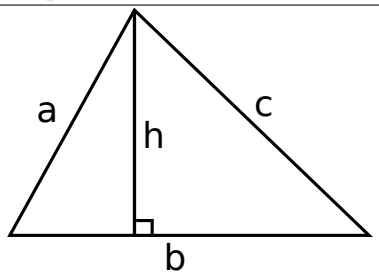
The bottom has length 14.60; top has length 9.13. The sides have lengths 8.23 and 8.61. The height is 7.94.

perimeter = _____

area = _____

Definition 4.1.12 Triangle. A **triangle** is a three sided shape. ◆

Table 4.1.13 Triangle

Shape	Perimeter	Area
	$a+b+c$	$\frac{1}{2}bh$

Because the three sides can all be different, the perimeter formula is just the sum of the lengths of each side. The **height** is a segment from a vertex down (perpendicularly) to the opposite side which is called the **base**. Any vertex/side pair can be used. Be aware that the height may not intersect the opposite side; consider Figure 4.1.15. The vertical, dashed line segments are heights for those two triangles. The one on the left is from the top vertex down to the bottom side. The one on the right is from the top vertex down to the extension (to the left) of the bottom side.

Example 4.1.14 What are the perimeter and area of the triangles in Figure 4.1.15?

The perimeter of the triangle on the left is

$$5.98 + 10.9 + 8.19 = 25.07 \approx 25.1.$$

The perimeter of the triangle on the right is

$$5.98 + 2.87 + 8.19 = 17.04 \approx 17.0.$$

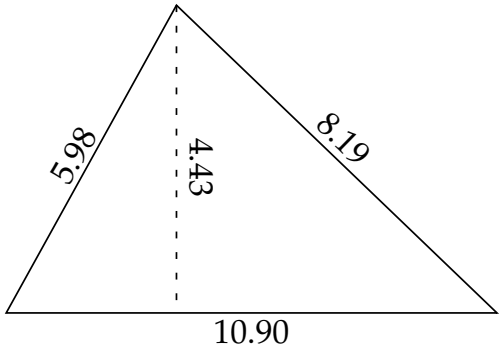
The area of a triangle, given in Table 4.1.13 is $\frac{1}{2}bh$. For the triangle on the left that is

$$\text{Area} = \frac{1}{2}10.90 \cdot 4.43 = 24.\underline{1}435 \approx 24.1.$$

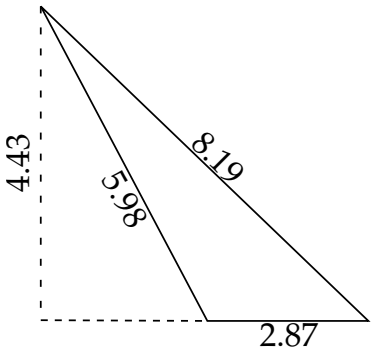
For the triangle on the right the area is

$$\text{Area} = \frac{1}{2}2.87 \cdot 4.43 = 6.\underline{3}5705 \approx 6.36$$

▲



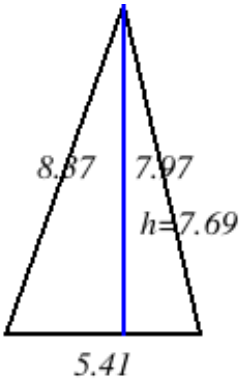
(a) Obtuse Triangle



(b) Acute Triangle

Figure 4.1.15 Two Triangles

Checkpoint 4.1.16



Find the perimeter and area of the triangle above. Side lengths are measurements.
perimeter = _____
area = _____

Checkpoint 4.1.17 In Table 4.1.13 the height is labeled h. Draw the other two heights.

Theorem 4.1.18 Pythagorean Theorem. *For a triangle containing a right angle*

$$a^2 + b^2 = c^2$$

where a and b are the lengths of the sides adjacent to the right angle and c is the third side.

Example 4.1.19 Consider the triangle in Figure 4.1.15(b). Consider the segments of length 4.43, 5.98, and the horizontal dashed segment. If we want to know the length of the dashed segment, we can use this theorem. 5.98 is the length of the side not adjacent to (opposite from) the right angle (c in the formula).

$$\begin{aligned} 4.43^2 + b^2 &= 5.98^2. \\ 19.\underline{6}249 + b^2 &= 35.\underline{7}604. \\ b^2 &= 16.\underline{1}355. \\ \sqrt{b^2} &= \sqrt{16.\underline{1}355}. \\ b &\approx 4.0\underline{1}690179 \\ &\approx 4.02. \end{aligned}$$

▲

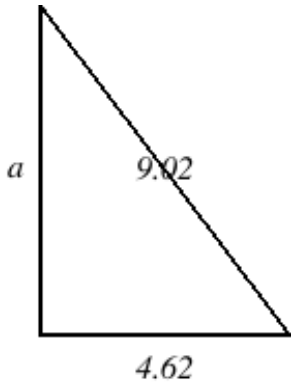
Example 4.1.20 Another use of this theorem is determining if a triangle is a right triangle. Notice that the triangle in Figure 4.1.15(a) appears to have a right angle at the top (opposite the side of length 10.90).

If this is a right triangle, then $a^2 + b^2$ will equal c^2.

$$\begin{aligned} a^2 + b^2 &= \\ 5.98^2 + 8.19^2 &= \\ 35.7604 + 67.0761 &= 102.8365. \\ c^2 &= 102.8365. \\ c &= \sqrt{102.8365} \\ &\approx 10.14083330 \\ &\approx 10.1. \end{aligned}$$

Because $10.1 \neq 10.9$ this is not a right triangle, but it appears to be close. ▲

Checkpoint 4.1.21



What is the length of the unmeasured side of the triangle? _____

In Section 7.3 we will develop a version of this statement for triangles without a right angle.

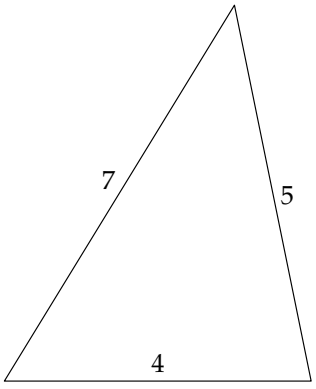
The formula for area of a triangle above requires that we be able to calculate the height. In Section 7.1 we will learn to calculate this height if we know an angle. This next formula enables us to calculate the area of a triangle without knowing any angles or the height.

Theorem 4.1.22 Heron's Formula. *The area of a triangle can be calculated using the three sides.*

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$.

Example 4.1.23 Calculate the area of a triangle with side lengths 4, 5, and 7.



According to Heron's formula

$$\begin{aligned} s &= \frac{1}{2}(4+5+7) \\ &= 8. \\ \text{Area} &= \sqrt{8(8-4)(8-5)(8-7)} \\ &= \sqrt{96} \\ &\approx 9.79795897 \\ &\approx 9.8. \end{aligned}$$

Rounding to one decimal place here is an arbitrary choice for this example. ▲

The next example illustrates tracking significant digits while using Heron's formula to calculate area.

Example 4.1.24 Calculate the area of the triangles in Figure 4.1.15.

According to Heron's formula for the triangle in Figure 4.1.15(a)

$$s = \frac{1}{2}(5.98 + 8.19 + 10.90)$$

Each term is precise to the 100ths (2 places)

$$\begin{aligned} &= 12.5\underline{3}5. \text{ Addition maintains to the 100ths (4 sigfigs)} \\ \text{Area} &\approx \sqrt{12.5\underline{3}5(12.5\underline{3}5 - 5.98)(12.5\underline{3}5 - 8.19)(12.5\underline{3}5 - 10.90)} \end{aligned}$$

Terms precise to 100ths, 3 sigfigs

$$\begin{aligned} &\approx \sqrt{12.5\underline{3}5(6.5\underline{5}5)(4.3\underline{4}5)(1.6\underline{3}5)} \text{ Product maintains 3 sigfigs} \\ &\approx \sqrt{58\underline{3}.7199977} \text{ Root maintains 3 sigfigs} \\ &\approx 24.\underline{1}6 \\ &\approx 24.2. \end{aligned}$$

For the triangle in Figure 4.1.15(b)

$$\begin{aligned} s &= \frac{1}{2}(5.98 + 8.19 + 2.87) \text{ Each term precise to 100ths} \\ &\approx 8.52. \text{ Sum maintains to the 100ths} \\ \text{Area} &\approx \sqrt{8.52(8.52 - 5.98)(8.52 - 8.19)(8.52 - 2.87)} \end{aligned}$$

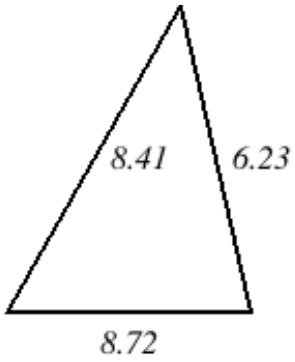
Sums maintain to the 100ths

$$\begin{aligned} &\approx \sqrt{8.52(2.54)(0.33)(5.65)} \text{ Smallest factor has 2 sigfigs} \\ &\approx \sqrt{4\underline{0}.34927160} \text{ Root maintains 2 sigfigs} \\ &\approx 6.\underline{3}5210765 \\ &\approx 6.4. \end{aligned}$$

▲

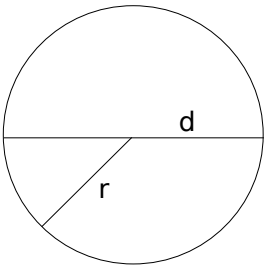
Notice that the results from Heron's formula are slightly different from the results using the $\frac{1}{2}bh$ formula. This reminds us that different calculations can result in different rounding. If the difference does not impact the application we do not care. If it does have a negative impact, then we must measure more precisely.

Checkpoint 4.1.25



What is the area of the triangle above? _____
Side lengths are measurements.

Table 4.1.26 Circle

Shape	Perimeter	Area
	$2\pi r$ πd	πr^2 $\pi \frac{d^2}{4}$

When performing calculations with π we will need to select an appropriate approximation. Because it is possible to obtain an approximation with arbitrary precision (as many decimal places as we want), we will select the approximation so that rounding is not affected.

Example 4.1.27 For a circle with radius 7.31 what are the perimeter and area?
The perimeter, given in Table 4.1.26, is $2\pi r$. For radius 7.31

$$\begin{aligned} \text{perimeter} &= 2\pi(7.31) \\ &\approx 2(3.14159)(7.31) \\ &= 45.\underline{9}300458. \\ &\approx 45.9. \end{aligned}$$

We need to approximate π to at least as many significant digits as the others numbers to avoid reducing precision. We could use more.

The area, given in Table 4.1.26, is πr^2. For radius 7.31

$$\begin{aligned} \text{area} &= \pi(7.31)^2 \\ &= (3.14159)(7.31)^2 \\ &\approx 16\underline{7}.8743174. \\ &\approx 168. \end{aligned}$$

▲

Example 4.1.28 What are the perimeter and area of a semi-circle with diameter 11.7?
The perimeter includes half the usual perimeter plus the length of the diameter. We use the diameter

version of the perimeter formula: πd.

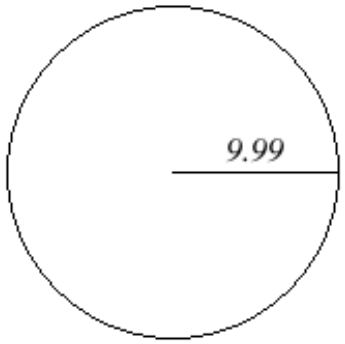
$$\begin{aligned}\text{perimeter} &= \frac{1}{2}\pi(11.7) + 11.7 \\ &= \frac{1}{2}(3.14159)(11.7) + 11.7 \\ &= 18.\underline{3}783015 + 11.7 \\ &= 30.\underline{0}783015 \\ &\approx 30.1.\end{aligned}$$

The area is simply half of the usual area.

$$\begin{aligned}\text{area} &= \frac{1}{2}\pi(11.7)^2 \\ &= \frac{1}{2}(3.14159)(11.7)^2 \\ &\approx 21\underline{5}.0261276. \\ &\approx 215.\end{aligned}$$

▲

Checkpoint 4.1.29



Find the perimeter and area of the circle above. The radius is a measurement.
perimeter = _____
area = _____

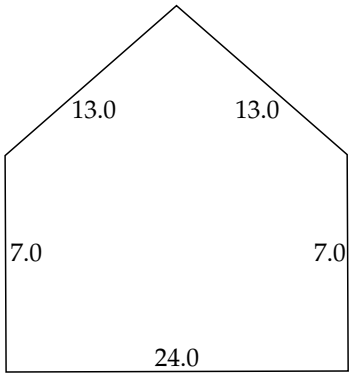
4.1.2 Applying Geometry

When we encounter geometric questions, they will often involve shapes consisting of more than one basic shape such as the park boundry above. Our first task then is to break down these shapes into pieces that are basic shapes (ones we already know). Then we can use the geometric properties to calculate results.

Note that often we can break down a complex shape in more than one way. Which way we choose will depend on what we are trying to calculate and what information we have available.

Example 4.1.30

(a) Calculate the area of this wall given the dimensions given in feet.



First we notice that we can describe the wall as a rectangle with a triangle on top of it. The side of the rectangle not drawn will also have length 24.0. This is also the base side of the triangle.

The sides of the rectangular area are 7.0 ft (height) and 24.0 ft (width). This means that area is 7.0 ft $\times$ 24.0 ft $= 1\bar{6}8$ ft^2.

The top is a triangle with two sides of length 13.0 and one of length 24.0. We do not know the height of the triangle so it will be easier to use Heron's formula for area.

$$\begin{aligned} s &= \frac{1}{2}(13.0 + 13.0 + 24.0) \\ &= 25.0. \\ \text{area} &= \sqrt{25.0(25.0 - 13.0)(25.0 - 13.0)(25.0 - 24.0)} \\ &= \sqrt{25.0(12.0)(12.0)(1.0)} \\ &= \sqrt{3600} \\ &= 6\bar{0}. \end{aligned}$$

The total area then is $1\bar{6}8 + 6\bar{0} = 228 \approx 230$ square feet.

(b) Calculate the perimeter of this wall given the dimensions given in feet.

There are five (5) edges. There sum is $7.0 + 24.0 + 7.0 + 13.0 + 13.0 = 64.0$ feet.

(c) What is the (tallest) height of the wall?

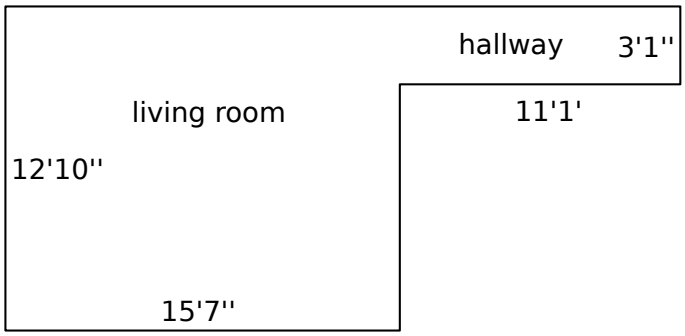
The height comes from the center of the wall (peak of the triangle). The height of the wall is the height of the rectangle (7.0 ft) plus the height of the triangle. We must calculate the height of the triangle. At this time we know the area and two area formulae. Height is part of the first area formula, and from Heron's formula we already know the area. Putting these two together gives us the following.

$$\begin{aligned} 6\bar{0} &= \frac{1}{2}24.0h. \\ 6\bar{0} &= 12.0h. \\ \frac{6\bar{0}}{12.0} &= h. \\ 5.0 &= h. \end{aligned}$$

The total height is then $7.0 + 5.0 = 12.0$.

▲

Example 4.1.31 Calculate the square footage of the combined living room and hallway.



This layout is rectangular. The living room is a rectangle with height 12 ft 10 in and width 15 ft 7 in. The hallway adds an additional rectangle of height 3 ft 1 in and width 11 ft 1 in.

To calculate square footage we need to convert everything to the same unit. To avoid rounding issues we will convert to inches then back to feet in the end.

The living room height is $12 \text{ ft} \cdot \frac{12 \text{ in}}{1 \text{ ft}} + 10 \text{ in} = 154 \text{ in}$. The width is $15 \text{ ft} \cdot \frac{12 \text{ in}}{1 \text{ ft}} + 7 \text{ in} = 187 \text{ in}$. The area is $(154 \text{ in})(187 \text{ in}) = 28798 \text{ in}^2$.

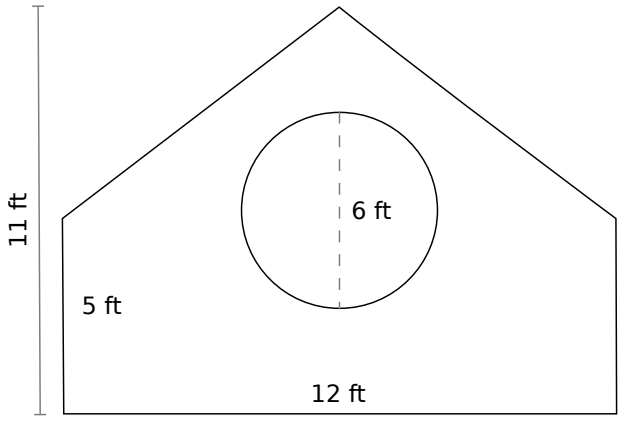
The hallway height is $3 \text{ ft} \cdot \frac{12 \text{ in}}{1 \text{ ft}} + 1 \text{ in} = 37 \text{ in}$. The width is $11 \text{ ft} \cdot \frac{12 \text{ in}}{1 \text{ ft}} + 1 \text{ in} = 133 \text{ in}$. The area is $(37 \text{ in})(133 \text{ in}) = 4921 \text{ in}^2$.

The total floor surface is $28798 \text{ in}^2 + 4921 \text{ in}^2 = 33312 \text{ in}^2$. We can now convert that back to feet.

$$33719 \text{ in}^2 \cdot \left(\frac{1 \text{ ft}}{12 \text{ in}} \right)^2 = \frac{33719}{144} \text{ ft}^2 \approx 234.1597 \text{ ft}^2 \approx 234 \text{ ft}^2.$$

We are using significant digits rounding. All measurements were accurate to one inch. If we converted to decimal feet we would end up with measurements like 3 ft 1in ≈ 3.0833 ft. The repeated decimal would have to be rounded whereas the inch measurement did not need to be rounded. Avoiding rounding avoids introducing error. ▲

Example 4.1.32 Calculate the area to be painted on this wall with a circular window.



The wall consists of a rectangle with a triangle on top. The rectangle has width 12 ft and height 5 ft for an area of $(5 \text{ ft})(12 \text{ ft}) = 60 \text{ ft}^2$.

We have the base of the triangle (12 ft), but we do not have the height of the triangle. However, we know that the total wall height is 11 ft, and the part of the height from the rectangal is 5 ft. This means that the rest is from the triangle which has height $(11 \text{ ft}) - (5 \text{ ft}) = 6 \text{ ft}$. Thus the area is $\frac{1}{2}(12 \text{ ft})(6 \text{ ft}) = 36 \text{ ft}^2$.

The area of the circle is $\pi \left(\frac{6 \text{ ft}}{2} \right)^2 \approx 28.27433389 \text{ ft}^2 \approx 28 \text{ ft}^2$.

The total wall area is therefore

$$60 \text{ ft}^2 + 36 \text{ ft}^2 - 28.27433389 \text{ ft}^2 \approx 67.72566611 \text{ ft}^2 \approx 68 \text{ ft}^2.$$

It is important to round up to have enough paint. ▲

The following example requires geometric thinking, but does not require any formula. Especially important is deciding on the type of rounding which, in this case, is determined by physical constraints.

Example 4.1.33 Katie is building a large scale abacus for a park. Her plan is to build it from treated 2x4 lumber. Her plan is shown in Figure 4.1.34 Note the depth of each piece of wood is 3.5". If you are wondering why a 2x4 is 1.5 in x 3.5 in, note that the nominal size (2x4 in this case) is based on the initial cut. The lumber shrinks as it cures and again when it is planed smooth.

Because we must have enough wood, we will round up all approximations. Measurements are accurate to an 8th of an inch.

(a) What is the total number of feet of lumber (2x4) needed?

Solution. There are two boards of length 60 inches and three boards of length 27 inches. The total length is

$$2(60 \text{ in}) + 3(27.0 \text{ in}) = 120 \text{ in}.$$

We convert this to feet using a ratio.

$$120 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} = 10 \text{ ft}$$

(b) If a standard 2x4 is 96.0 inches long, what is the smallest number of boards Katie can purchase to have enough lumber?

Solution. If a 60 inch section is cut from a 96 inch board, we have $96 \text{ in} - 60 \text{ in} = 36 \text{ in}$ left. This is long enough for one of the 27 inch segments but not more. Thus two boards will cover all but the last 27 inch segment. We need three (3), 96 inch boards.

(c) If the boards are painted before they are assembled, what is the total surface area of the boards to be painted? Paint cans are rated for the number of square feet they can cover. We will want the solution in units of square feet.

Solution. Each board has six surface. Each surface size appears twice (e.g., top and bottom). For the long segments these areas are

$$\begin{aligned} 60 \text{ in} \cdot 3.5 \text{ in} &= 210 \text{ in}^2, \\ 60 \text{ in} \cdot 1.5 \text{ in} &= 90 \text{ in}^2, \\ 1.5 \text{ in} \cdot 3.5 \text{ in} &= 5.25 \text{ in}^2. \end{aligned}$$

For the short segments these are

$$\begin{aligned} 27 \text{ in} \cdot 3.5 \text{ in} &= 94.5 \text{ in}^2, \\ 27 \text{ in} \cdot 1.5 \text{ in} &= 40.5 \text{ in}^2, \\ 1.5 \text{ in} \cdot 3.5 \text{ in} &= 5.25 \text{ in}^2. \end{aligned}$$

Thus the total area is

$$\begin{aligned} &2(2)(210 \text{ in}^2) + 2(2)(90 \text{ in}^2) + 2(2)(5.25 \text{ in}^2) \\ &+3(2)(94.5 \text{ in}^2) + 3(2)(40.5 \text{ in}^2) + 3(2)(5.25 \text{ in}^2) = \\ &\qquad 810 + 360 + 21 + 567 + 246 + 31.5 \text{ in}^2 = 2035.5 \text{ in}^2 \end{aligned}$$

Finally we convert this to square feet using the method of Example 1.1.23.

$$\begin{aligned} 2035.5 \text{ in}^2 \cdot \left(\frac{1 \text{ ft}}{12 \text{ in}}\right)^2 &= \\ \frac{2035.5 \text{ in}^2}{12^2 \text{ in}^2} \text{ ft}^2 &= \end{aligned}$$

$$\frac{2035.5}{144}\ \text{ft}^2 \approx 14.13541667\ \text{ft}^2.$$
$$\approx 15.$$

We round up because we want to be safe and because we likely cannot order an amount to cover exactly 14.1 square feet.

(d) What is the area that is hidden, that is, cannot be seen after assembling?

Solution. This would be where the three short boards touch the long boards. There are six places where this happens which are all the same shape $3.5\text{ in} \cdot 1.5\text{ in} = 5.25\text{ in}^2$. The covered surface is on both the short boards and the matching spot on the long boards, so there are 12 of these surfaces for $12 \cdot 5.25\text{ in}^2 \approx 63\text{ in}^2$.

▲

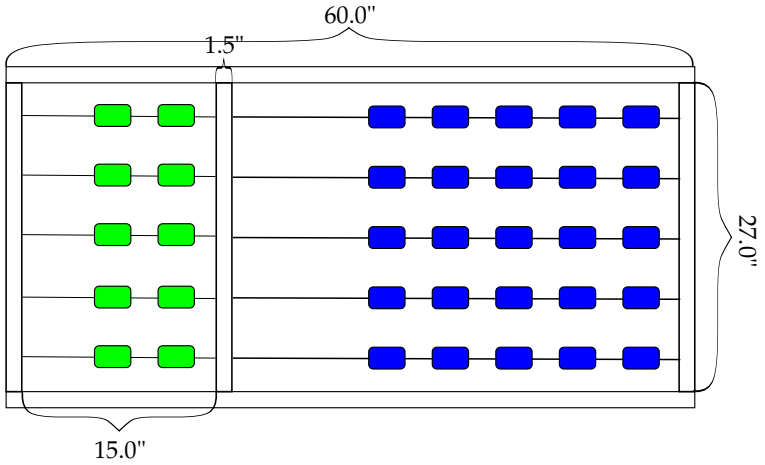
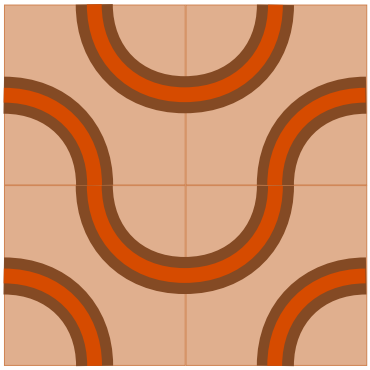


Figure 4.1.34 Abacus

The following example also illustrates rounding according to physical constraints.

Example 4.1.35 We are purchasing carpet squares to carpet a rectangular room that is 15.5 ft by 19.0 ft. These carpet squares are 18 in by 18 in. A box contains 10 squares. How many boxes of carpet squares do we need purchase? This image shows four carpet squares together.



We need to know how many square across and down the room. For the number of squares across we can divide the width of the room (15.5 ft) by the width of a carpet square (18 in). This requires matching units, so $15.5\text{ ft} \cdot \frac{12\text{ in}}{\text{ft}} = 186\text{ in}$ Thus the number of squares across is $\frac{186}{18} = 10.333$. This means the final square will be a third of a square in width. While it might seem efficient to reuse the rest of that carpet square in the rows above, we cannot because the pattern would not continue correctly. Thus we simply round up to 11 squares across.

Next we calculate the number of squares down. $19.0 \text{ ft} \cdot \frac{12 \text{ in}}{\text{ft}} = 228$ in. Thus the number of squares down is $\frac{228}{18} = 12.666$. Again to have enough squares we must round up to 13 squares down.

Thus the total number of carpet squares needed is $11 \cdot 13 = 143$. Because each box has 10 squares each, we need $\frac{143}{10} = 14.3$ boxes. To have enough we need 15 boxes. In practice we would likely buy at least one extra box in case there are problems. ▲

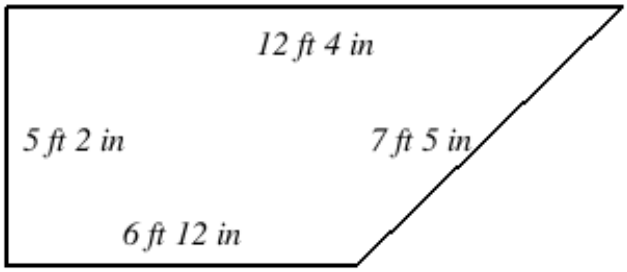
Example 4.1.36 At first the following might seem to be a reasonable way to solve this problem. First, calculate the total square inches of floor. Using the conversion to inches from above $186 \cdot 228 = 42408$ square inches of floor.

Next we can divide by the number of square inches each carpet squares covers to count the number of squares. Each square covers $18^2 = 324$ square inches. Thus we calculate that we need $\frac{42408}{324} = 130.888$ carpet squares.

This is substantially fewer than the 143 we calculated previously. The reason is this calculation assumes we can use each of the partial squares somewhere else in the room. This is true when the pattern does not need to match, but not in this case.

Now you may understand why such repeating patterns are avoided in most carpets. ▲

Checkpoint 4.1.37



What is the area of this floor in square feet? ____

4.1.3 Quantitative Literacy

We use many methods to help us remember formulae and to use the correct one. We can use dimensional analysis (watching units) to help us use the correct formula in geometry.

Example 4.1.38 A circle has radius $r = 5.3$ in. To calculate the area of the circle we will use either the formula $2\pi r$ or πr^2. One is area, and the other is circumference (perimeter). If we recall that area has *square* units then we notice πr^2 will end up with $(5.3 \text{ in})^2$ which has squared units. Thus πr^2 is the area formula. $2\pi r$ must be circumference. ▲

Next we consider if there is any obvious relationship between the circumference and area of a circle. Specifically we ask if one is always bigger than the other.

Example 4.1.39 Is the area of a circle is always a larger number than the circumference of a circle? This might seem reasonable because the circumference is just the edge of the circle and area is all of it. Guesses are an important part of mathematics. After we guess, we check our conjecture. Consider the following.

$$\begin{aligned} r &= 1. \\ 2\pi(1) &\approx 6.2832. \\ \pi(1)^2 &\approx 3.1416 \\ 2\pi(2) &\approx 12.5664. \\ \pi(2)^2 &\approx 12.5664 \\ 2\pi(3) &\approx 18.8496. \\ \pi(3)^2 &\approx 28.2743 \end{aligned}$$

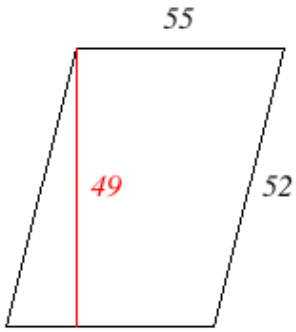
Immediately we can see that the circumference is neither always less or always more than the area. If $r = 1$, circumference is a larger number than the area. For $r = 3$ the relationship is reversed.

Generally, there is no relationship between perimeter and area of shapes. ▲

4.1.4 Exercises

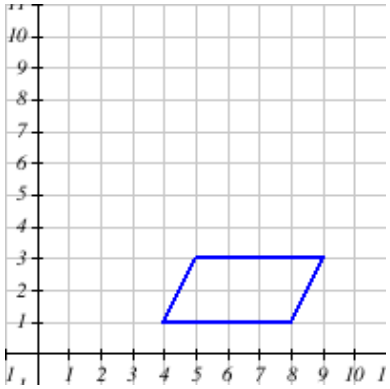
Practice Formulae. Use the appropriate formulae to calculate requested values.

1. Find the area of the parallelogram shown below.



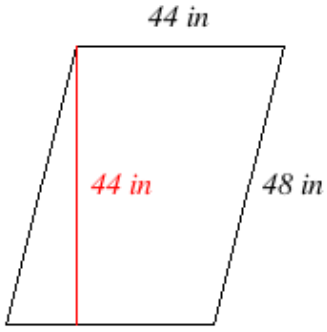
_____

2. Find the area of the parallelogram shown below.



Area = _____

3. Find the perimeter and area of the parallelogram shown below.



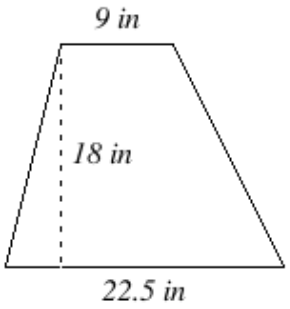
The perimeter is _____

1. square inches
2. cubic inches
3. inches

The area is _____

1. inches
2. square inches
3. cubic inches

4.

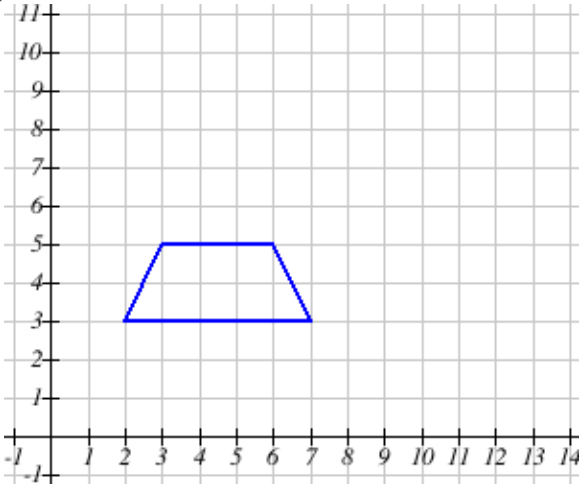


Find the area of the trapezoid pictured above.

___

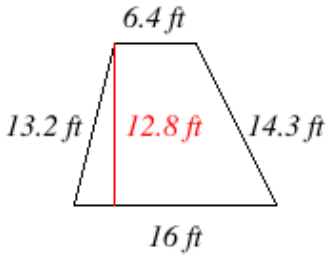
1. in
2. sq in

5. Find the area of the trapezoid shown below.



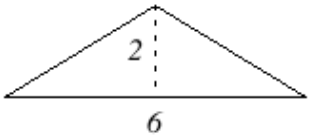
Area = _____

6. Find the perimeter and area of the trapezoid shown below. Round your answer to the nearest hundredth as needed.



The perimeter is _____ feet.
The area is _____ square feet.

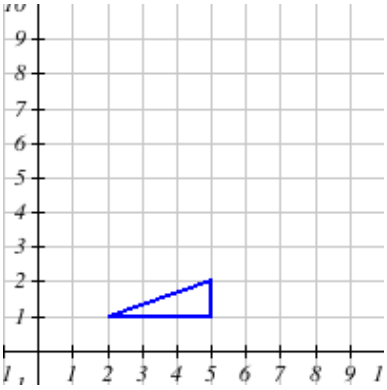
7.



Find the area of the triangle pictured above.

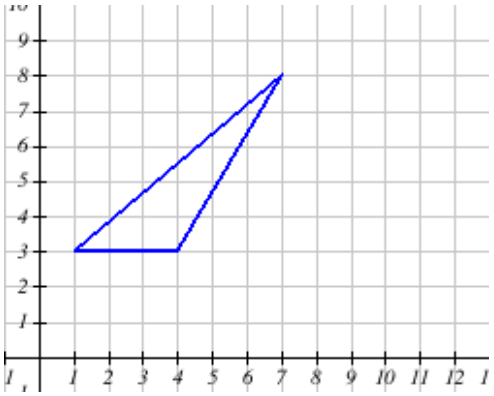
_____

8. Find the area of the triangle shown below.



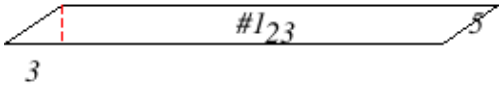
Area = _____

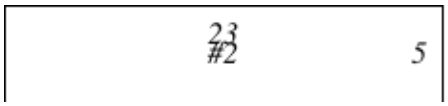
9. Find the area of the triangle shown below.

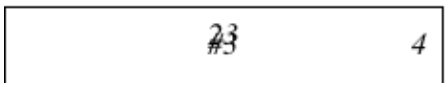


Area = _____

10. Below are three quadrilaterals

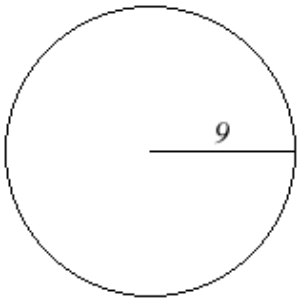






What is the perimeter and area of the 1st quadrilateral?
Perimeter =_____ Area =_____
What is the perimeter and area of the 2nd quadrilateral?
Perimeter =_____ Area =_____
What is the perimeter and area of the 3rd quadrilateral?
Perimeter =_____ Area =_____

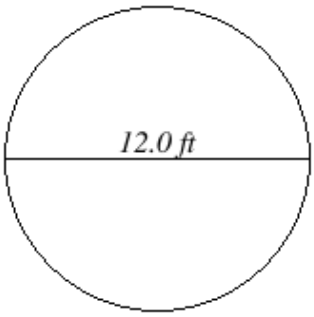
11.



Find the area of the circle pictured above.

_____

12.



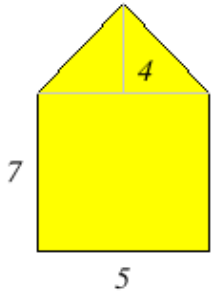
Find the area of the circle pictured above.

Round your answer using the rules of working with significant figures. Note using the approximation of 3.14 for π may result in an incorrect answer.

_____ ft^2

Compound Shapes. Use one or more formulas to calculate the areas shown.

13.

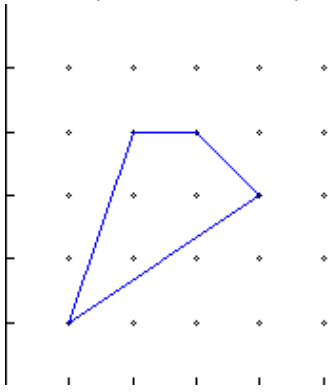


Find the area of the shaded region above.

_____

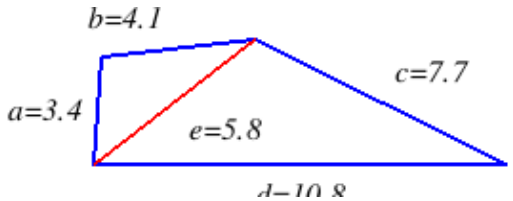
14. What is the area of the quadrilateral on the Geoboard below?

The distance between points vertically and horizontally is 1 cm.



The area is ___ cm^2.

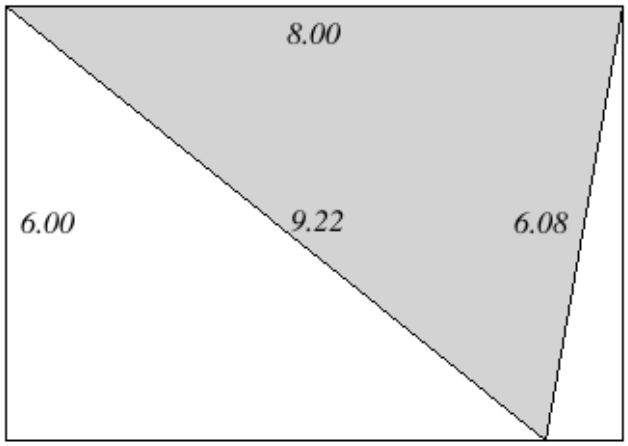
15.



Find the area of the quadrilateral shown. The lengths of the four sides (counterclockwise) are $a = 3.4$, $b = 4.1$, $c = 7.7$, and $d = 10.8$, with diagonal length $e = 5.8$ from the intersection of a and d to the intersection of b and c . Measurements are in yards. What is the area of this figure?

area = _____ yd^2

16. The diagram below shows a triangle inscribed in a rectangle. The rectangle has width 8.00 in and height 6.00 in. One side of the triangle is a long side of the rectangle. The other triangle sides have lengths 6.08 in and 9.22 in.



What is the area of the shaded, inscribed triangle? ____ in^2

What is the area of the rectangle? ____ in^2

What is the area of the unshaded parts of the rectangle? ____ in^2

17. In the figure below, the angles that appear to be right angles *are in fact* right angles. Lengths in

the figure are given in units of centimeters.

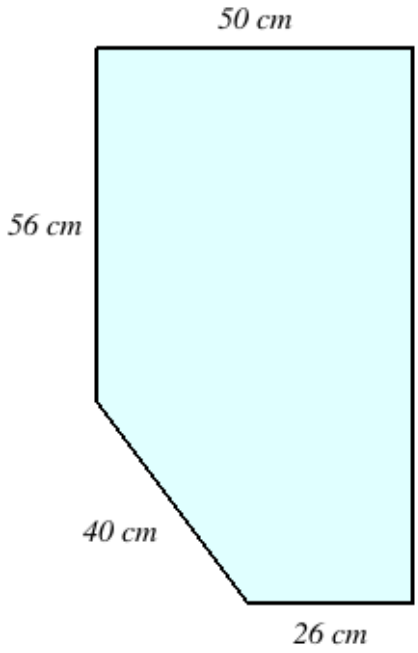


Image DescriptionA closed shape with five straight sides, with its interior shaded, where four of its five sides are either horizontal or vertical. Beginning at the top right corner and moving to the left, the path around the shape starts with a side that is labeled 50 centimeters, then moving straight down, a side is labeled 56 centimeters, then moving diagonally down and right, a side is labeled 40 centimeters, then moving to the right, a side is labeled 26 centimeters, and finally moving straight up, a side is unlabeled, connecting back to the original starting point at top right.

What is the *area* of the shaded shape? Give your answer as a decimal number, and then choose the correct units from the dropdown menu.

Answer: _____

1. centimeters
2. cubic centimeters
3. square centimeters

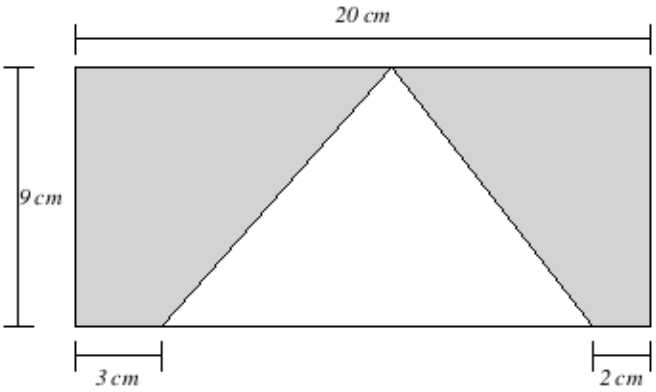
What is the *perimeter* of the shaded shape? Give your answer as a decimal number, and then choose the correct units from the dropdown menu.

Answer: _____

1. cubic centimeters
2. centimeters
3. square centimeters

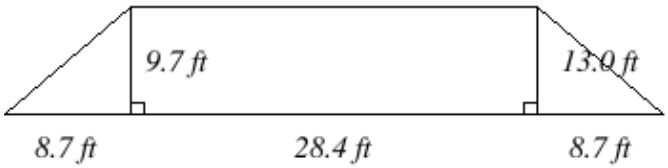
18. Find the area of the shaded region. Recall these formulas. Area of a rectangle: $A = lw$. Area of a

triangle: $A = \frac{1}{2}bh$.



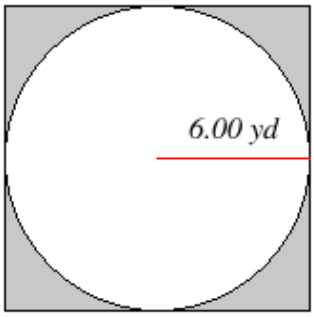
Area = _____

19. Find the perimeter and area of the following composite figure. Round using the rules for measurements.



Perimeter _____ feet
Area _____ square feet

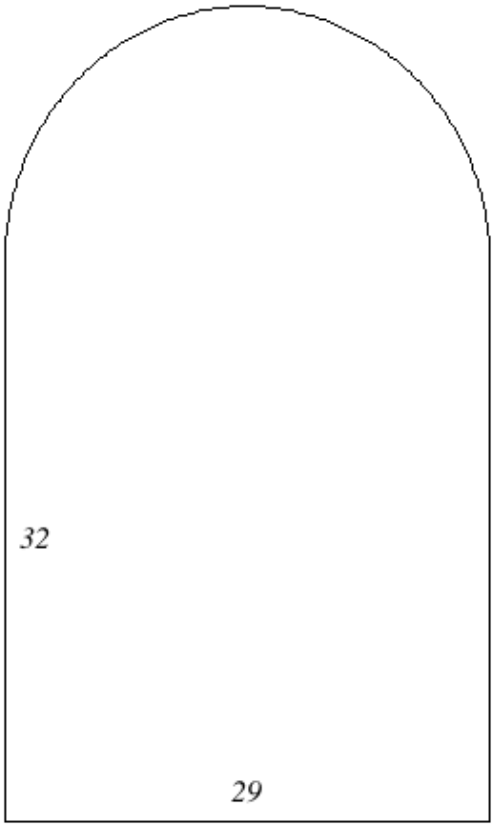
20.



This is a circle with radius 6.00 yd inscribed in a square with no side labeled. Round using the rules for measurements.

1. What is the area of the square? ___ yd^2
2. What is the area of the circle? ___ yd^2
3. What is the area of the shaded section? ___ yd^2

21.



What is the area of the figure above? _____

22. A rounded rectangle is a rectangle where the corners are replaced with quarter circle arcs. If we know the height and width of the rectangle and the radius of the corners, we can calculate the area.



If the height is 10 cm, the width is 17 cm, and the corner radius is 2 cm, what is the total area?

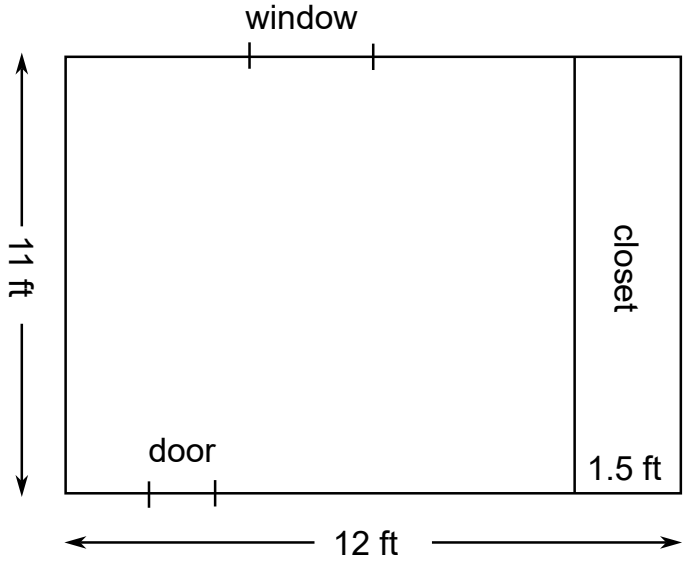
_____

Round to the nearest tenth.

Hint: There are multiple ways to break down this problem into known parts.

Geometry in Applications. Identify shapes and use appropriate formulae to calculate the results requested.

23. Londell wishes to carpet a rectangular room. He will not carpet the floor inside the closet.



Which is the number of square yards, to the nearest square yard, of carpet needed to carpet the floor of the room? Carpet is sold in units of square yards so round to ensure you have enough.

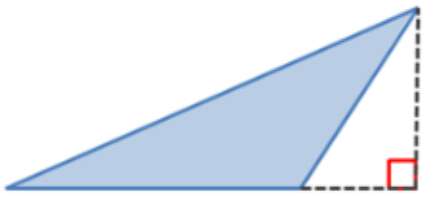
_____

24. A triangular parcel of land has sides of lengths 360. feet, 860. feet and 601 feet.

1. What is the area of the parcel of land? Round your answer to appropriate significant digits.
   Area = _____ ft^2
2. If land is valued at \$1700 per acre (1 acre is 43,560 ft^2), what is the value of the parcel of land? Base on rounded area. Round your value to the nearest dollar.
   value =\$ _____

25. Chee needs enough mulch to cover the triangular garden shown and enough paving stones to border it. If one bag of mulch covers 11 square feet and one paving stone provides a 6-inch border, how many bags of mulch and how many stones does she need to buy?

Figure is not to scale



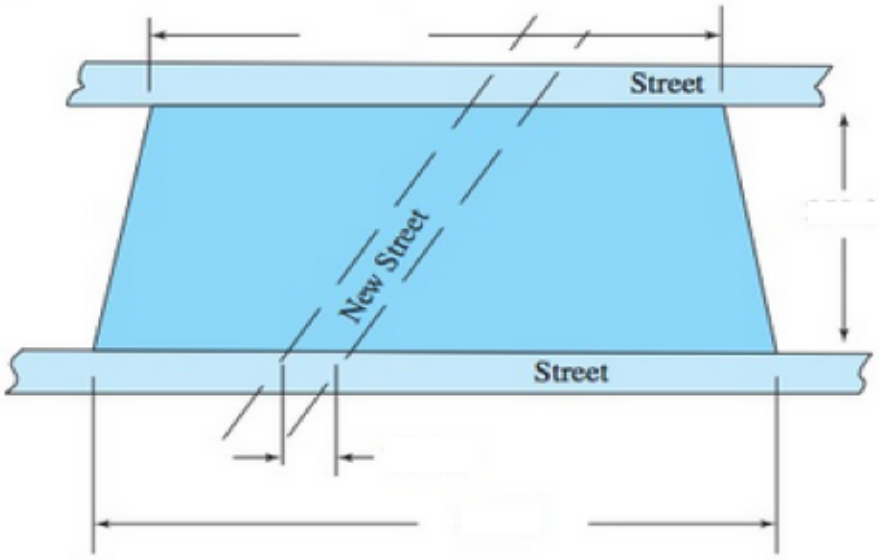
Area: _____ ft^2 _____ bags of mulch
Perimeter: _____ ft _____ paving stones

26. Ashley has a sprinkler than covers a circular area with diameter 12 feet. What is the area of lawn that the sprinkler covers? (Use 3.14 for π)

_____ square feet

27. A parcel of land lies between two parallel streets. How many acres will remain in the parcel after

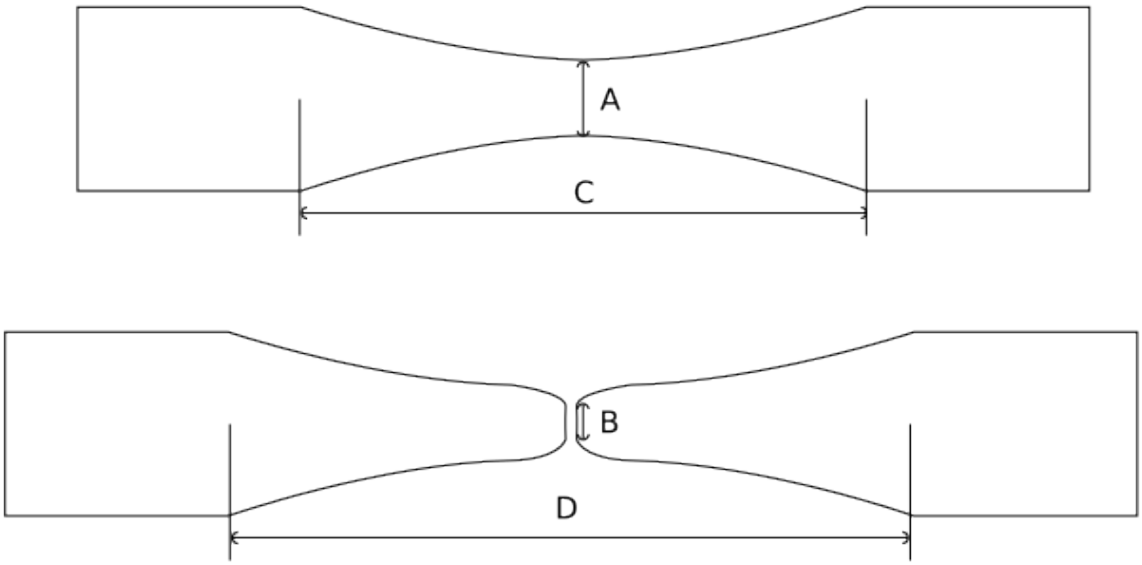
the strip shown is taken for a new street? (1 acre = 43,560 ft^2)



_____ acres

Quantitative Literacy. Use you knowledge of geometry to determine what makes sense.

28.



One measure of ductility, the ability of metal to be pulled into wires, is area reduction. This is measured as the percent decrease of area after pulling a sample piece of metal. The image above illustrates the metal sample before (on top) and after the pull (on bottom). A is the area of the round sample before the pull. B is the area after the pull (in this case pulled apart). The formula is $R = \dfrac{A - B}{A} \times 100$.

Guido uses callipers to measure the diameters and calculates the areas $A = \pi\dfrac{(d_A)^2}{4}$ and $B = \pi\dfrac{(d_B)^2}{4}$, where d_A and d_B are the diameters. He uses these to calculate the area reduction. Wolfgang calculates the area reduction using the same formula but solely the diameters (effectively $A = d_A$ and $B = d_B$).

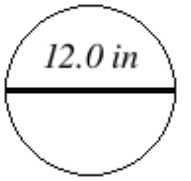
Do they end up with the same result for ductility?

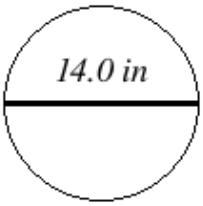
1. no
2. yes

Why is that the case?

29. Rachael decides to order a pizza and wants to determine whether a medium, which costs \$15.99, or a large, which costs \$18.99 is the better deal.

Here are the medium (left) and large (right) pizzas.





The diameter of the large pizza is ____ times as large as the medium pizza.

What is the area of the medium pizza? Round your answer to the nearest tenth. ____ in^2

What is the area of the large pizza? Round your answer to the nearest tenth. ____ in^2

The area of the large pizza is ____ times as large as the medium pizza.

If the medium pizza costs \$15.99 , how much should the large pizza cost to have each square inch of pizza be of equal value? Round your answer to the nearest cent. \$ ____

Recall that the costs of the pizzas were \$15.99 and \$18.99. Which pizza is the better deal? (The better deal has a more pizza per dollar.)

1. The large pizza.
2. The medium pizza.
3. They are the same price per square inch.
4. This can't be determined.

30. Find the area of a triangle with sides *27*, *65*, and *37*

_____

Your answer should be accurate to the tenths place.

Enter DNE if the triangle cannot exist.

4.2 Geometric Reasoning 3D

This section addresses the following topics.

- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Identify shapes and apply their properties (skill)

This section presents geometric properties, illustrates identifying shapes in applications, and illustrates breaking down complex shapes into simple ones.

4.2.1 Formulae

This section defines the two properties of interest and provides the formulae for some common shapes. Memorizing all of the formulae is not likely useful: in a job you will be able to look them up. However, anything you use a lot (e.g., prisms and spheres) is worth memorizing.

Use this section to learn to identify which measurement is used in each part of a formula and how to determine those from diagrams or descriptions of shapes. The next section will illustrate using these properties in applications.

The **surface area** of a 3D shape is the cummulative area of all the 2D areas of the shape. You can think of it as the amount of paint needed to cover the object. The **volume** of a 3D shape is a measure of what it takes to fill a 3D shape. In the figure below showing a paper cone for holding popcorn, the surface area is the amount of wax needed to coat the inside of the cone so the popcorn (butter) does not soak through the paper. The volume is how much popcorn can be held.

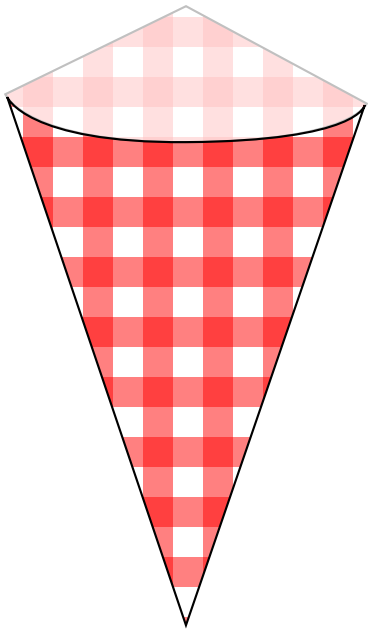


Figure 4.2.1 Complex Shape with Surface and Volume

Rounding. How we round on geometric problems will depend as always on the application. For example a carpenter will round to units that can be measured with a tape measure like 1/16 of an inch. Contextless examples in this section will be rounded using significant digits. When using significant digits treat all numbers in the formulae as exact numbers.

Definition 4.2.2 Prism. A **prism** is a solid consisting of two identical polygons connected by parallelograms. ◆

These look like a polygon has been extruded. That polygon is called the **base**. If the base is extruded perpendicular to its surface, then it is called a **right prism**. Typically this means the sides are rectangles.

Prisms are named for their base shape. For example, there are triangular prisms and pentagonal prisms. The figure below shows four prisms. Those on the left are right prisms, and those on the right are skewed. The top two are triangular prisms, and the bottom two are trapezoidal prisms.

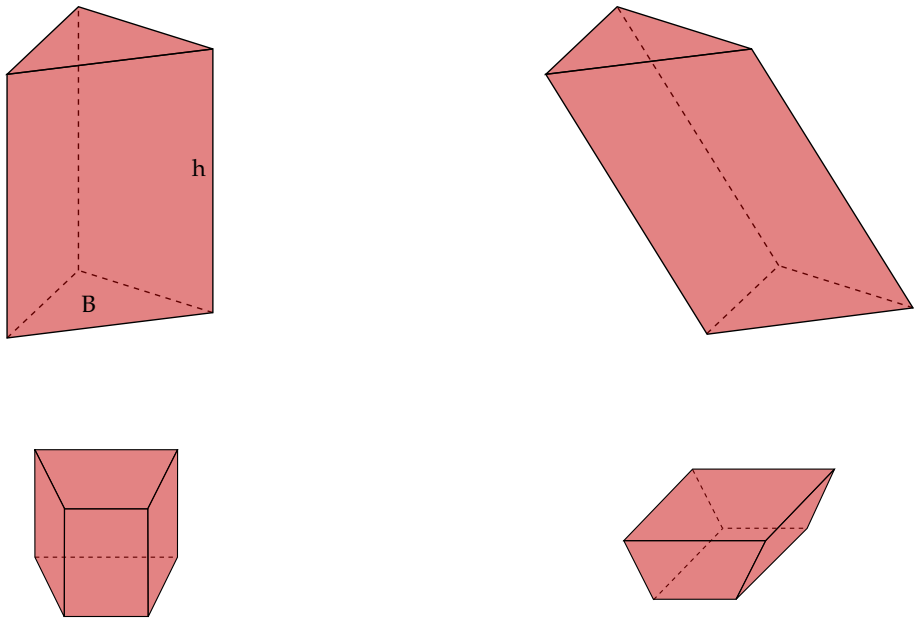
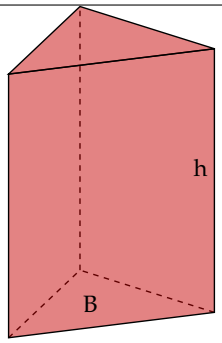


Figure 4.2.3 Various Prisms

Table 4.2.4 Prisms

Shape	Surface Area	Volume
	sum of area of all sides	$V = Bh$

For surface area the sides are always parallelograms (rectangles for right prisms). For the volume B is the area of the base (the triangle or trapezoid in the four examples). h is the height or distance from the bottom to the top (or front to back if we tip the prism over). Note this is not along an edge, but perpendicular to the base.

Example 4.2.5

(a) What is the surface area of the right triangular prism in Figure 4.2.6?

The surface area consists of the areas of the two right triangles and the three rectangles. Because the triangles are right triangles, they both have area $\frac{1}{2} \cdot 2.12\underline{5} \cdot 3.12\underline{5} = 3.3\underline{2}03125$. The rectangle areas are $2.12\underline{5} \cdot 10.25\underline{0} = 21.7\underline{8}125$ and $3.12\underline{5} \cdot 10.25\underline{0} = 32.0\underline{3}125$. For the third rectangle we need to calculate the side length. Because this is a right triangle we can use the Pythagorean theorem.

$$(2.12\underline{5})^2 + (3.12\underline{5})^2 = c^2.$$

$$4.51\underline{5}625 + 9.76\underline{5}625 = c^2. \qquad \text{Squaring maintains 4 sigfigs}$$

$$14.28\underline{1}25 = c^2. \qquad \text{Adding precise to 1000ths}$$

$$\sqrt{14.28\underline{1}25} = c. \qquad \text{Root maintains 5 sigfigs}$$

$$3.77905 \approx c.$$
$$3.7791 \approx c.$$

The third rectangle has area $3.77905 \cdot 10.250 \approx 38.73\underline{5}2625$. The total area is

$$2(3.3\underline{2}03125) + 21.7\underline{8}125 + 32.0\underline{3}125 + 38.73\underline{5}2625 \approx 99.1\underline{8}83875$$
$$\approx 99.19.$$

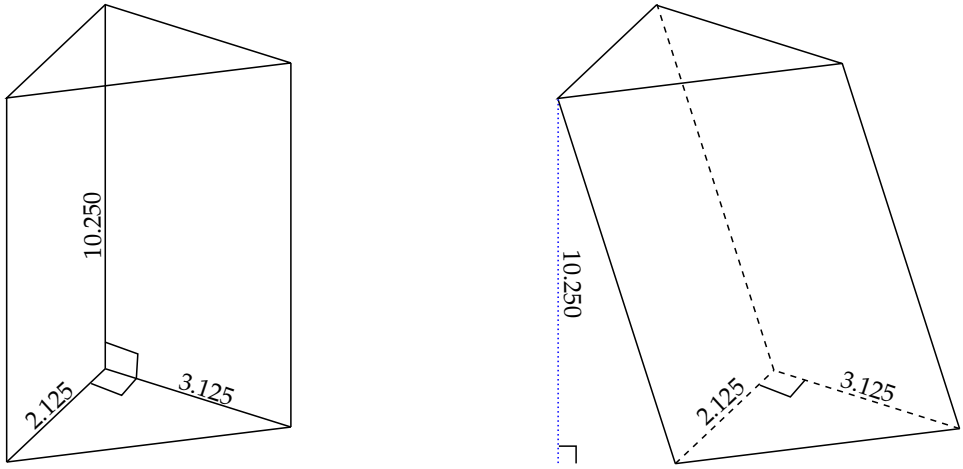
(b) What is the volume of the triangular prism in Figure 4.2.6(a)?

The volume is the area of the triangle, 3.320, times the height of the prism, 10.250. Thus the area is $3.320 \cdot 10.250 = 34.03$.

(c) What is the volume of the triangular prism in Figure 4.2.6(b)?

The volume is the area of the triangle, 3.320, times the height of the prism, 10.250. Thus the area is $3.320 \cdot 10.250 = 34.03$. Notice that the volume is not affected by the tilt of the prism. This is not true of the surface area which we do not have sufficient information to calculate here.

▲

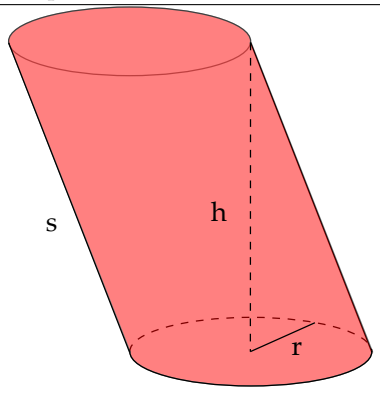


(a) Right Triangular Prism **(b)** Non-Right Triangular Prism

Figure 4.2.6 Calculate the surface area and volume

Definition 4.2.7 Cylinder. A **cylinder** is a circular prism. ◆

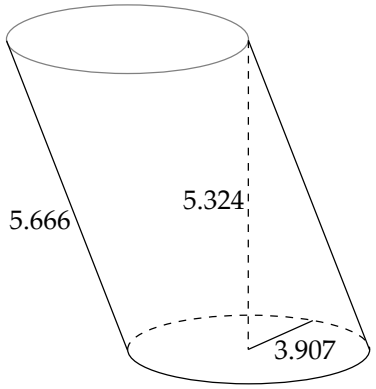
Table 4.2.8 Cylinder

Shape	Lateral Surface Area	Volume
	$A = 2\pi r \cdot s$	$V = \pi r^2 h$

The height (h) of the cylinder is the distance from the top disc down to the plane in which the bottom disc sits. It does not necessarily touch both discs (cylinder skewed so far the discs do not overlap). If the cylinder is not slanted then $s = h$. Notice the volume is not dependent on the angle of slant but the lateral surface area is. To obtain the complete surface area add the area of the top and bottom discs to the lateral surface area.

The formula for the area of the side of a cylinder looks like width ($2\pi r$) times height (s) which is the formula for area of a rectangle. Indeed if you think about a can (the type that contain paint, soup, or other substances), the label if peeled off is a rectangle. The height of that rectangle is the height of the can and the width of that rectangle is enough to wrap around the can.

Example 4.2.9 What are the total surface area and volume of this slanted cylinder?



The lateral surface area is

$$2\pi(3.90\underline{7})(5.66\underline{6}) \approx$$
$$2(3.1415\underline{9})(3.90\underline{7})(5.66\underline{6}) \approx 139.\underline{0}911452.$$

We must approximate π precise to at least as many positions as the rest of the numbers. Otherwise we will decrease the overall precision. We do not need to use more positions: the extra disappears in the rounding at the end. Here we use the portion most commonly memorized or listed in a table.

The area of the top and bottom are both

$$\pi(3.90\underline{7})^2 \approx$$
$$(3.1415\underline{9})(3.90\underline{7})^2 \approx 47.9\underline{5}526865.$$

Thus the total surface area is

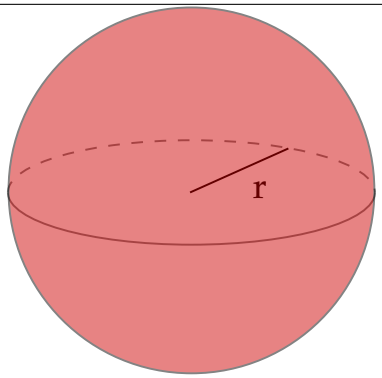
$$2(47.9\underline{5}526865) + 139.\underline{0}911452 \approx 235.\underline{0}016825 \approx 235.0.$$

The volume is

$$\begin{aligned} V &= \pi(3.90\underline{7})^2(5.32\underline{4}) \\ &\approx (3.1415\underline{9})(3.90\underline{7})^2(5.32\underline{4}) \\ &\approx 255.\underline{3}138503 \\ &\approx 255.3 \end{aligned}$$

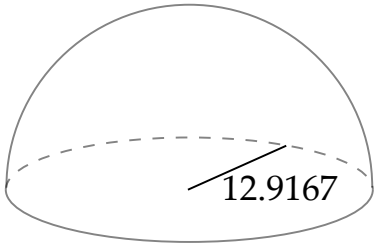
▲

Table 4.2.10 Sphere

Shape	Surface Area	Volume
	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$

The 4 and 4/3 are exact numbers.

Example 4.2.11 What are the surface area and volume of this spherical dome? The dome is sitting on the ground so the only surface area of importance is the top surface.



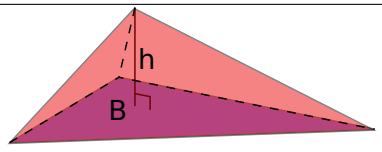
Because this is a dome (half sphere) we divide the formula by 2.

$$\begin{aligned} A &= \frac{1}{2}(4\pi(12.9167)^2) \text{ Squaring maintains 6 sigfigs} \\ &\approx \frac{1}{2}(4(3.14159)(12.9167)^2) \text{ Approximate to 6 sigfigs} \\ &\approx 1048.2\underline{9}2907 \text{ Product maintains 6 sigfigs} \\ &\approx 1048.29 \end{aligned}$$

$$\begin{aligned} V &= \frac{1}{2}\left(\frac{4}{3}\pi(12.9167)^3\right) \\ &\approx \frac{1}{2}\left(\frac{4}{3}(3.14159)(12.9167)^3\right) \\ &\approx 4513.4\underline{9}4997 \\ &\approx 4513.49 \end{aligned}$$

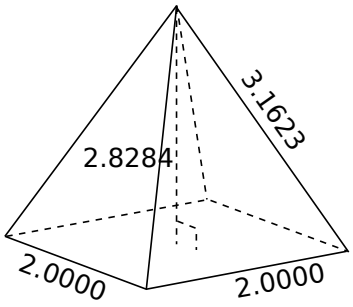
▲

Table 4.2.12 Pyramid

Shape	Surface Area	Volume
	sum of surfaces	$V = \frac{1}{3}Bh$

Example 4.2.13

(a) What is the surface area of this pyramid?



This pyramid consists of four, identical triangular sides and a square base. Note the side lengths are given as exact numbers.

The area of one of the triangles can be found using Heron's formula.

$$\begin{aligned} s &= \frac{1}{2}(2.0000 + 3.1623 + 3.1623) \\ &= 4.1623. \text{ All numbers precise to 4 decimal places.} \\ A &= \sqrt{s(s-3.1623)(s-3.1623)(s-2.0000)} \end{aligned}$$

Subtraction maintains 4 decimal places

$$\begin{aligned} &= \sqrt{(4.1623)(1.0000)(1.0000)(2.1623)} \text{ Product will have 5 sigfigs} \\ &= \sqrt{9.000\underline{1}4129} \text{ Root will maintain 5 sigfigs} \\ &= 3.000\underline{0}23548 \\ &= 3.0000. \end{aligned}$$

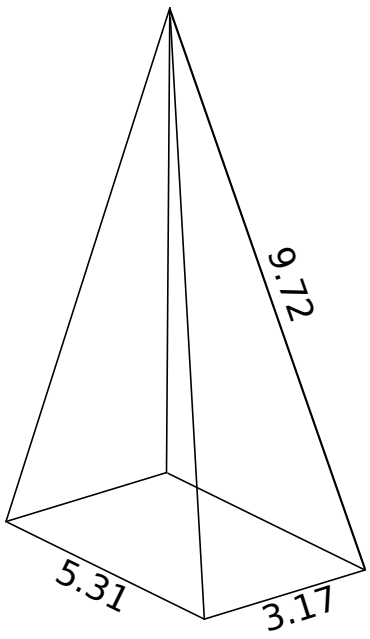
The area of the base is $2.0000 \cdot 2.0000 = 4.0000$. Thus the surface area is $4.0000 + 4(3.0000) = 16.0000$.

(b) What is the volume of this pyramid?

The volume is $V = \frac{1}{3}(2.0000^2)(2.8284) = 3.7712$.

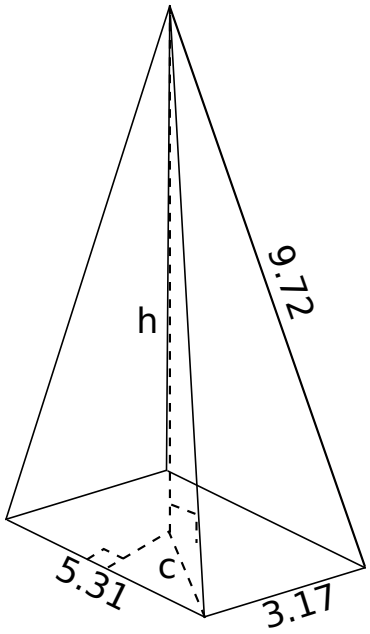
▲

Example 4.2.14 Calculate the volume of this rectangular pyramid.



To calculate the volume we need the height which is not labeled. This is probably because we cannot put a measuring tape along the height (interior). However, we can calculate the height based on the right triangle with the height as one leg and the edge length as the hypotenuse. The second leg is in the base from the center to the corner of the pyramid.

We do not have the length of the segment on the base (center to corner). If we add a line from the center of the base to the midpoint of one of the sides, we form another right triangle with the desired length as the hypotenuse. The legs are each half the length of one of the sides. The Pythagorean theorem will give us the hypotenuse length.



$$\left(\frac{5.31}{2}\right)^2 + \left(\frac{3.17}{2}\right)^2 = c^2.$$

$7.0\underline{4}9025 + 2.5\underline{1}2225 = c^2.$ Square maintains 3 sigfigs

$$\begin{aligned} 9.5\underline{6}125 &= c^2. && \text{Addition maintains precision to 2 places} \\ \sqrt{9.5\underline{6}125} &= c. && \text{Root maintains 3 sigfigs} \\ 3.0\underline{9}21271 &\approx c. \\ 3.09 &\approx c. \end{aligned}$$

Next, we can use this length and the given side length to calculate the height.

$$\begin{aligned} 3.0\underline{9}21271^2 + h^2 &= 9.72^2. \\ 9.5\underline{6}125 + h^2 &= 94.\underline{4}784. && \text{Square maintains 3 sigfigs} \\ h^2 &= 94.\underline{4}784 - 9.5\underline{6}125. \\ h^2 &= 84.\underline{9}1715. \end{aligned}$$

Subtraction maintains precision to only one place

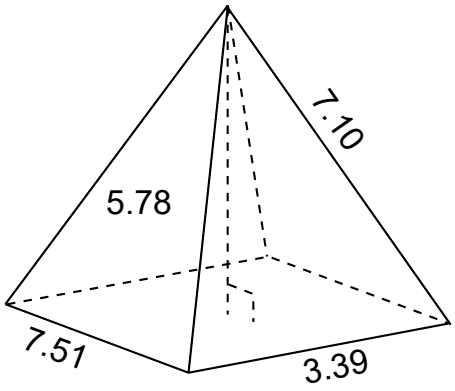
$$\begin{aligned} h &= \sqrt{84.\underline{9}1715}. && \text{Root maintains 3 sigfigs} \\ h &\approx 9.2\underline{1}505019 \\ h &\approx 9.22. \end{aligned}$$

Finally we can calculate the volume. The base has area $5.31 \cdot 3.17 = 16.\underline{8}327 \approx 16.8$. This pyramid has volume

$$V = \frac{1}{3}(16.\underline{8}327)(9.2\underline{1}505019) \approx 51.\underline{7}0472510 \approx 51.7.$$

▲

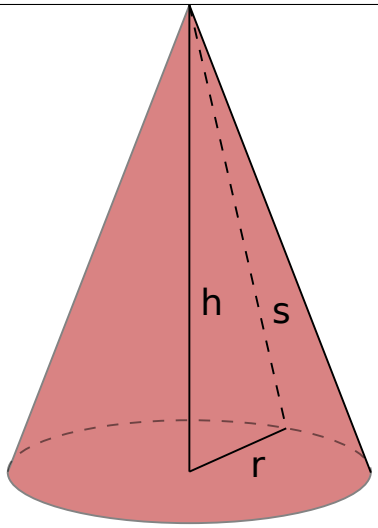
Checkpoint 4.2.15



For the pryamid above (not to scale) determine the
Surface area (all): _____
Volume: _____

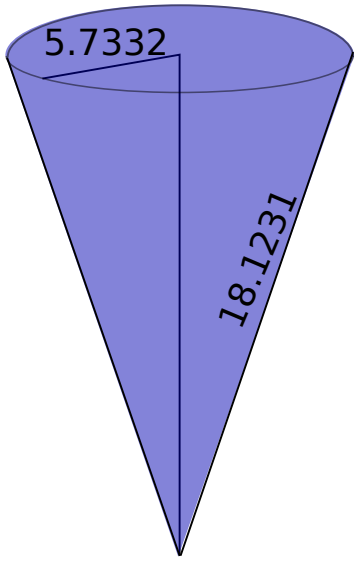
Checkpoint 4.2.16 What is the relationship between the volume of a pyramid to the volume of a prism with the same base? "same base" means the same shape and size. The pyramid could then be placed inside the prism.

Table 4.2.17 Cone

Shape	Lateral Surface Area	Volume
	$A = \pi r\sqrt{h^2 + r^2}$ or $A = \pi rs$	$V = \frac{\pi}{3}r^2h$

The surface area formulas apply only when it is a right circular cone (tip of the cone directly above the center of the circle). For the full surface area we add the area of the circle at the bottom of the cone. The volume formula works for all cones.

Example 4.2.18 Calculate the total surface area and the volume of this cone.



The surface area can be calculated directly. The lateral surface area is

$$\begin{aligned} LA &= \pi rs \\ &\approx (3.14159)(5.7332)(18.1231) \text{ Minimum 5 sigfigs} \\ &\approx 326.4\underline{2}17471 \\ &\approx 326.42 \end{aligned}$$

The area of the top circle is

$$\begin{aligned} TA &= \pi r^2 \\ &\approx (3.14159)(5.7332)^2 \text{ Minimum 5 sigfigs} \end{aligned}$$

$$\approx 103.2\underline{6}27509$$
$$\approx 103.26$$

The total area therefore is $A = 326.4\underline{2}17471 + 103.2\underline{6}27509 \approx 429.6\underline{8}44980 \approx 429.68$.

To calculate the volume we need to first calculate the height. Looking at the image we see that the labeled radius and side length are part of a right triangle with the height. This means

$$\begin{aligned} h^2 + 5.7332^2 &= 18.1231^2 \text{ Squaring maintains sigfigs} \\ h^2 + 32.86\underline{9}5224 &\approx 328.44\underline{6}7536 \text{ Precise to 3 places} \\ h^2 &\approx 328.44\underline{6}7536 - 32.86\underline{9}5224 \\ h^2 &\approx 295.57\underline{7}1714 \\ h &\approx 17.192\underline{3}5794 \text{ Root preserves 6 sigfigs} \\ &\approx 17.1924. \end{aligned}$$

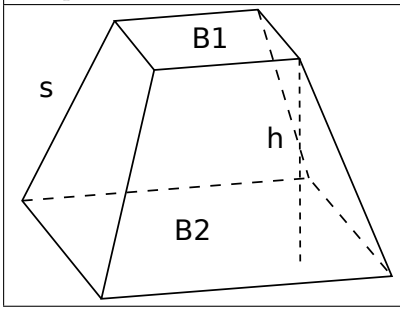
Having calculated the height we can calculate the volume

$$\begin{aligned} V &= \frac{\pi}{3} r^2 h \\ &\approx \frac{3.14159}{3}(5.7332)^2(17.192\underline{3}5794) \\ &\approx 591.7\underline{7}67250 \\ &\approx 591.78 \end{aligned}$$

▲

The frustums are compound shapes: they are obtained by subtracting one shape from itself. The latin word **frustum** means "cut off." This is the etymology of "frustrated" which refers to a cut off hope. For these formulae the bases must be parallel. Note P_i below refers to the perimeter of the bases.

Table 4.2.19 frustum of a Pyramid

Shape	Lateral Surface Area	Volume
	$\frac{1}{2}s(P_1 + P_2)$	$V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$

P_1 is the perimeter of the lower base (bottom polygon). It will be the sum of the lengths of each side. P_2 is calculated similarly for the upper base (top polygon). B_1 is the area of the lower base (bottom polygon). It will be calculated based on the type of polygon (e.g., area of a triangle, parallelogram, etc.). B_2 is the area of the upper base.

Example 4.2.20

(a) What is the surface area of the frustum of the pyramid in Figure 4.2.21?

The surface area consists of the areas of the base and top which are triangles and the three sides which are rhobmi. For the base and top we know the lengths of the three sides, so we can use Heron's formula to calculate their areas. The perimeter of the lower base (triangle) is $3(3.500) = 10.500$ (The number of sides, 3, is an exact number so precision is not changed.) The perimeter of the upper base (triangle)

is $3(2.200) = 6.600$.

$$\begin{aligned}
s_1 &= 10.500. \\
A_1 &= \sqrt{10.500(10.500-3.500)^3} \\
&= \sqrt{10.500(7.000)^3} \\
&= \sqrt{360\underline{1}.5} \\
&\approx 60.0\underline{1}24987 \\
&\approx 60.01. \\
s_2 &= 6.600. \\
A_2 &= \sqrt{6.600(6.600-2.200)^3} \\
&= \sqrt{6.600(4.400)^3} \\
&= \sqrt{562.\underline{2}144} \\
&\approx 23.7\underline{1}106071 \\
&\approx 23.71.
\end{aligned}$$

The three sides are rhombi. The area of one of them is

$$\begin{aligned}
A &= \frac{1}{2}(2.422)(3.500+2.200) \\
&= \frac{1}{2}(2.422)(5.700) \text{ Remains precise to 3 places} \\
&= 6.90\underline{2}7 \text{ Maintains 4 sigfigs} \\
&\approx 6.903.
\end{aligned}$$

Thus the surface area of the entire shape is $60.0\underline{1}24987 + 23.7\underline{1}106071 + 3(6.90\underline{2}7) \approx 104.4\underline{3}16594 \approx 104.43$.

(b) What is the volume of the frustum of a pyramid in Figure 4.2.21?

We need the height (labeled) and the area of the two bases which were calculated in the previous step. Using these values

$$\begin{aligned}
V &= \frac{1}{3}(2.333)(60.01+23.71+\sqrt{(60.01)(23.71)}) \text{ Product maintains 4 sigfigs} \\
&= \frac{1}{3}(2.333)(60.01+23.71+\sqrt{142\underline{2}.8371}) \text{ Root maintains 4 sigfigs} \\
&= \frac{1}{3}(2.333)(60.01+23.71+37.7\underline{2}051299) \text{ All terms precise to 2 places} \\
&= \frac{1}{3}(2.333)(121.4\underline{4}0513) \text{ Product maintains 4 sigfigs} \\
&= 94.4\underline{4}023895 \\
&\approx 94.44.
\end{aligned}$$

▲

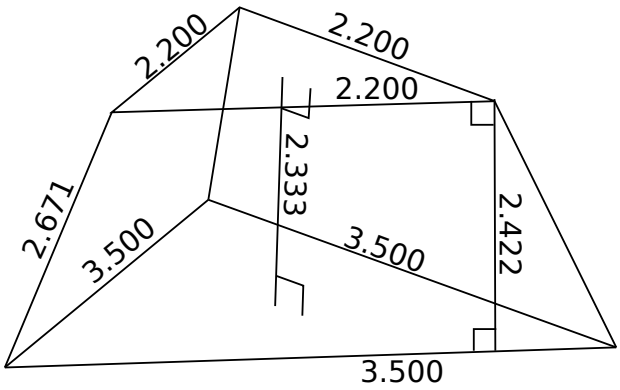
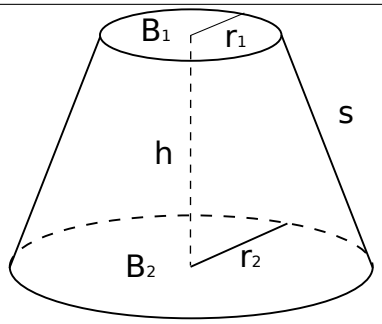


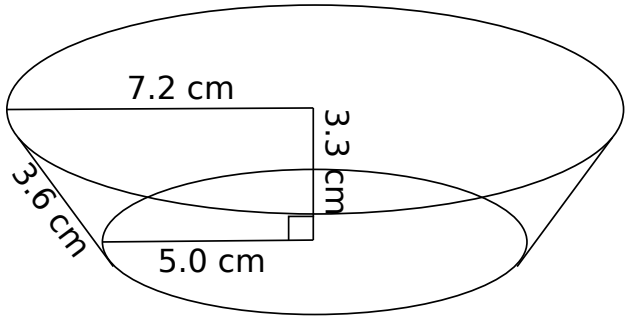
Figure 4.2.21 Calculate Area and Volume of this frustum

Table 4.2.22 frustum of a Cone

Shape	Lateral Surface Area	Volume
	$\pi s(r_1 + r_2)$	$V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$

The total surface area adds the area of the two discs (bottom and top) to the lateral surface area. The volume is the same formula as for a frustum of a pyramid because these are pyramids built on a circle.

Example 4.2.23 What is the surface area an volume of one part of a muffin tin? The surface area would be the area of a paper liner (assuming it is flat and not corrugated) and the volume is the amount of batter that can be held (if fully filled).



The surface area will be the lateral surface area plus the area of the bottom base (muffin tin is open on top).

$$\begin{aligned} LA &= \pi s(r_1 + r_2) \\ &\approx (3.14159)(3.6)(5.0 + 7.2) \text{ Sum maintains 1 place precision} \\ &\approx (3.14159)(3.6)(12.2) \text{ Product maintains 2 sigfigs} \\ &\approx 1\underline{3}7.9786328 \\ &\approx 140. \\ BA &= \pi r^2 \end{aligned}$$

$$\begin{aligned}
&\approx (3.14159)(5.0)^2\\
&\approx 7\underline{8}.53975\\
&\approx 79.
\end{aligned}$$

The total surface area is $1\underline{3}7.9786328 + 7\underline{8}.53975 \approx 2\underline{1}6.5183828 \approx 220$ cm squared.

The volume requires calculating the area of the top surface as well.

$$\begin{aligned}
TA &= \pi r^2\\
&\approx (3.14159)(7.2)^2\\
&\approx 1\underline{6}2.8600256\\
&\approx 160.
\end{aligned}$$

Now we can calculate the volume of the muffin tin.

$$\begin{aligned}
V &= \frac{1}{3}\left(3.3)(78+160+\sqrt{(78)(160)}\right) \text{ Product maintains 2 sigfigs}\\
&= \frac{1}{3}(3.3)\left(78+160+\sqrt{1\underline{2}480}\right) \text{ Root maintains 2 sigfigs}\\
&\approx \frac{1}{3}(3.3)(78+160+1\underline{1}1.7139) \text{ Root maintains 2 sigfigs}\\
&\approx \frac{1}{3}(3.3)(3\underline{4}9.7139) \text{ Product maintains 2 sigfigs}\\
&= 3\underline{8}4.6853\\
&= 380
\end{aligned}$$

The volume is 380 square centimeters. ▲

If we have the slant height (length of the side of a frustum of a cone) we can calculate the height. This is similar to the calculation of height in Example 4.2.14.

Example 4.2.24 Suppose the larger radius is 7.32, the smaller radius is 5.44, and the slant height is 6.09. The slant height is the hypotenuse of a right triangle with the height being one other side. The third side is on the larger base and has length $7.32 - 5.44 = 1.88$. We can use the Pythagorean theorem to calculate the height.

$$\begin{aligned}
h^2 + 1.88^2 &= 6.09^2. \text{ Square maintains 3 sigfigs}\\
h^2 + 3.5\underline{3}44 &= 37.\underline{0}881.\\
h^2 &= 37.\underline{0}881 - 3.5\underline{3}44. \text{ Subtraction precise to 1 place}\\
h^2 &= 33.\underline{5}537.\\
h &= \sqrt{33.\underline{5}537} \text{ Root maintains 3 sigfigs}\\
&\approx 5.7\underline{9}255567\\
&\approx 5.79.
\end{aligned}$$

▲

Checkpoint 4.2.25 For the surface area of a frustum of a cone we use the slant height: the distance from the edge of the bottom base to the edge of the top base. Here we consider limitations on that length.

For these questions suppose the bottom base has radius 5. The top base will vary depending on the slant height.

(a) If the height is 3, can the slant height be 2?

(b) What is the bottom of the range of possible slant heights for this frustum?

(c) If the height is 3 and the top base has radius 1, what is the slant height?

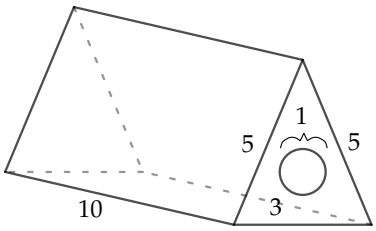
(d) If the height is 3 and the top base has radius 0.5, what is the slant height?

(e) What is the top of the range of possible slant heights for this frustum? Note this is for bottom base radius 5, height 3, and top base radius unrestricted (but smaller than bottom base).

4.2.2 Applying Geometry

Effectively using knowledge of geometric properties requires recognizing the shapes we know in problems we encounter and figuring out missing parameters from the parameters we do know. The latter we did in Example 4.2.14 and Example 4.2.24. We will extend our ability to calculate missing parameters in Chapter 7. In the following problems we identify shapes and apply the appropriate formulae.

Example 4.2.26 What is the total, external area of this birdhouse? All measurments are in inches and accurate to 1/8 inch.



First, we identify the shapes. The front and back are triangles. The front triangle has a circular hole in it. The bottom and two sides are rectangles. Although this is a 3D object surface area uses 2D formulae.

For the triangles, we have edge lengths and not the height, because it is easier to measure the edges than the middle. For the area we will use Heron's formula.

$$\begin{aligned} s &= \frac{1}{2}(5+5+3) \\ &= \frac{13}{2} \\ &= 6.5 \\ TA &= \sqrt{(6.5)\,(6.5-3)\,(6.5-5)^2} \\ &= \sqrt{(6.5)\,(3.5)\,(1.5)^2} \\ &= \sqrt{51.1875} \\ &\approx 7.1545 \end{aligned}$$

For the hole we know the diameter so we use the area formula $\frac{\pi}{4}d^2$. This hole has area

$$\frac{\pi}{4}(1)^2 \approx \frac{3.14159}{4} \approx 0.7853975$$

The area of the front triangle is $7.1545 - 0.7853975 \approx 6.3691025$.

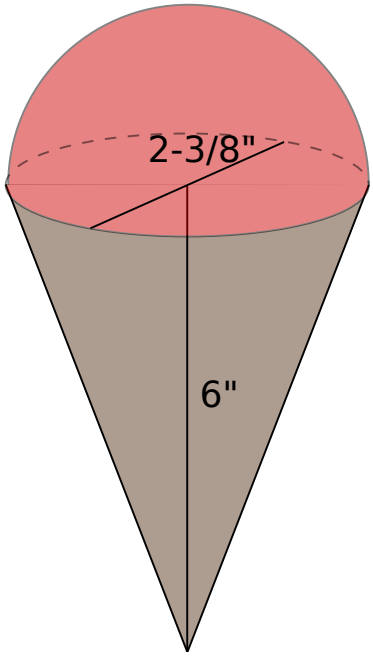
The bottom rectangle has area $3 \cdot 10 = 30$ square inches. The two sides both have area $5 \cdot 10 = 50$ square inches.

Thus the total area is

$$7.1545 + 6.3396025 + 2(50) + 30 \approx 143.5236025 \approx 143.6.$$

We round up because this is the safer estimate (better to have a little more than not enough for construction). ▲

Example 4.2.27 An ice cream cone has the dimensions shown below.



(a) What is the volume of the ice cream? No person can scoop with sufficient accuracy for a fraction of a square or cubic inch, so we will round to units.

Solution. First, we identify the shapes. This is a right circular cone with a half sphere on top. The diameter of the cone and sphere is $2\ \frac{3}{8} = 2.375$ inches. The radius then is half of that or $1\ \frac{3}{16} = 1.1875$. We convert to decimal inches, because in the end we would want to convert cubic inches (volume) to cups or pints or similar volume measures. This is easier with decimal inches.

The volume of the cone is

$$\begin{aligned} V_c &= \frac{\pi}{3}(1.1875)^2(7) \\ &\approx \frac{3.14159}{3}(1.1875)^2(7) \\ &\approx 10.33697647. \end{aligned}$$

The volume of the half sphere is

$$\begin{aligned} V_s &= \frac{1}{2}\cdot\frac{4}{3}\pi(1.1875)^3 \\ &\approx \frac{2}{3}(3.14159)(1.1875)^3 \\ &\approx 3.507188445. \end{aligned}$$

Thus the total volume of the ice cream is $10.33697647 + 3.507188445 \approx 13.84416492 \approx 14$ cubic inches.

(b) What is the surface area of the ice cream? Note the greater the area the faster it melts.

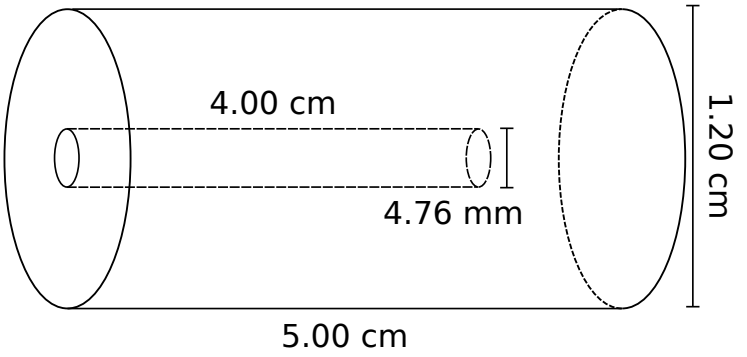
Solution. This is a right circular cone with a half sphere on top. The surface area of the cone is

$$\begin{aligned} A_c &= \pi(1.1875)\sqrt{6^2 + (1.1875)^2} \\ &\approx (3.14159)(1.1875)\sqrt{6^2 + (1.1875)^2} \\ &\approx (3.14159)(1.1875)\sqrt{37.410156} \\ &\approx (3.14159)(1.1875)(6.116384) \\ &\approx 22.81801626. \end{aligned}$$

The surface area of the half sphere is $\frac{1}{2} \cdot 4\pi(1.1875)^2 \approx 8.860265547$. Thus the total surface area of the ice cream is $22.81801626 + 8.860265547 \approx 31.67828181 \approx 32$ square inches.

▲

Example 4.2.28 A part is produced by drilling a cylindrical hole into a cylinder as shown below. If the weight of aluminum is 2.70g/cm^3 what is the total weight of this part? These are precision measurements; round using significant digits.



First, we calculate the volume of the cylinder. It has a radius of $1.20/2 = 0.60$ cm. The volume is

$$\begin{aligned} V_C &= \pi(0.60 \text{ cm})^2(5.00 \text{ cm}) \\ &\approx 3.14159(0.60 \text{ cm})^2(5.00 \text{ cm}) \\ &\approx 5.654862 \text{ cm}^3 \\ &\approx 5.7 \text{ cm}^3. \end{aligned}$$

Next, we need to calculate the volume of the hole that is drilled out. That radius is given in mm, so we should convert to cm first. 4.76 mm is 0.476 cm, because there are 10 millimeters per centimeter. It has a radius of $0.476/2 = 0.238$ cm. The volume is

$$\begin{aligned} V_C &= \pi(0.238 \text{ cm})^2(4.00 \text{ cm}) \\ &\approx 3.14159(0.238 \text{ cm})^2(4.00 \text{ cm}) \\ &\approx 0.7118088958 \text{ cm}^3 \\ &\approx 0.712 \text{ cm}^3. \end{aligned}$$

The total volum is $5.654862 - 0.7118088958 \approx 4.943053104 \approx 4.9$.
Finally, we can convert the volume to weight. The units suggest we can multiply,

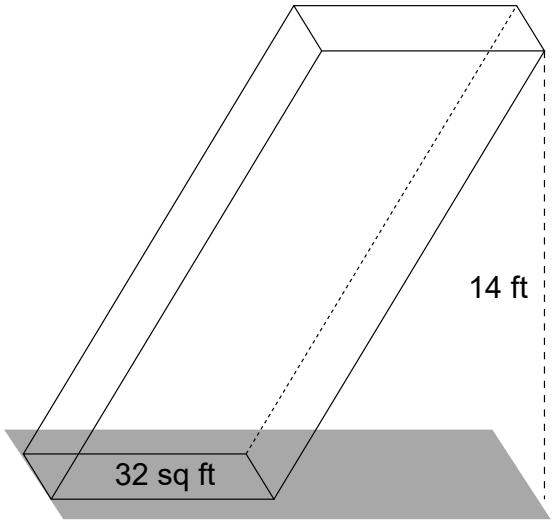
$$4.943053104 \text{ cm}^3 \cdot \frac{2.70 \text{ g}}{\text{cm}^3} \approx 13.34624338 \text{ g} \approx 13 \text{ g}.$$

▲

4.2.3 Exercises

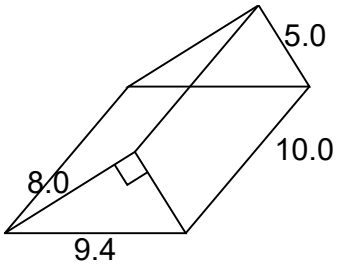
Practice Formulae. Use the appropriate formulae to calculate requested values.

1.



Calculate the volume.
$V =$ _____ ft^3

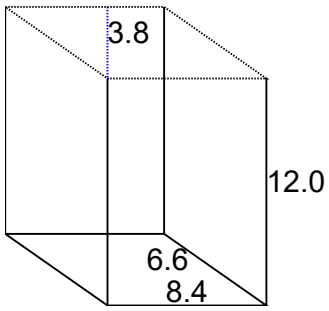
2.



One side is a right triangle with side lenths: 5.0, 8.0, and 9.4. This is extruded a length of 10.0. Round results according to the rules for measurements.

What is the volume? _____
What is the surface area? _____

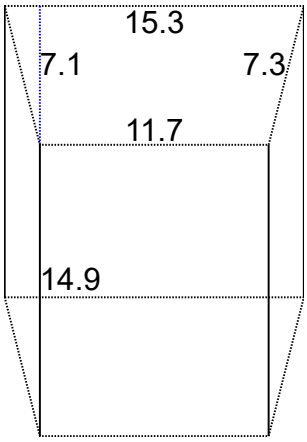
3.



One side is a parallelogram with side lengths: 6.6 and 8.4. The height of the parallelogram is 3.8. This parallelogram is extruded a length of 12.0. Round results according to the rules for measurements.

What is the volume? _____
What is the surface area? _____

4.

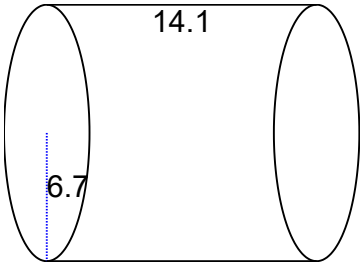


One side is a trapezoid with parallel sides having lengths: 11.7 and 15.3. The height of the trapezoid is 7.1. This trapezoid is extruded a length of 14.9. Round results according to the rules for measurements.

What is the volume? _____

What is the surface area? _____

5.

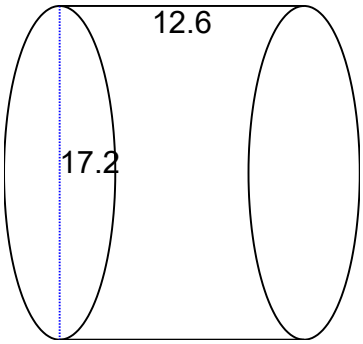


This is a right cylinder with radius 6.7 and length 14.1. Round results according to the rules for measurements.

What is the volume? _____

What is the surface area? _____

6.

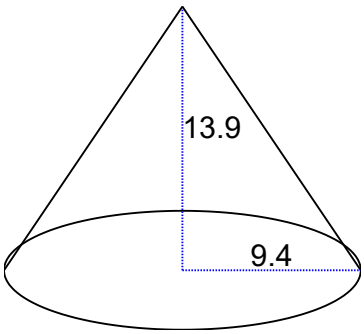


This is a right cylinder with diameter 17.2 and length 12.6. Round results according to the rules for measurements.

What is the volume? _____

What is the surface area? _____

7.

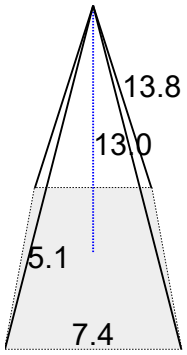


This is a right cone with radius 9.4 and height 13.9. Round results according to the rules for measurements.

What is the volume? _____

What is the surface area? _____

8.

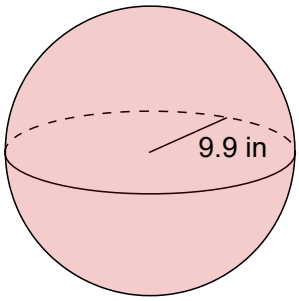


One side is a rectangle with side lengths: 5.1 and 7.4. The height of the pyramid is 13.0 and the long sides are length 13.8. Round results according to the rules for measurements.

What is the volume? _____

What is the surface area? _____

9.

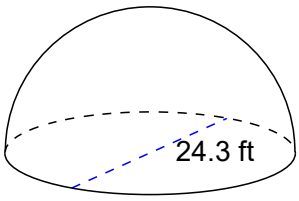


What is the volume of the sphere? _____ in^3

What is the surface area of the sphere _____ in^2

Round according to the rules for measurements.

10.



What is the volume of the sphere? _____ ft^3

What is the surface area of the sphere _____ ft^2

Round according to the rules for measurements.

11.

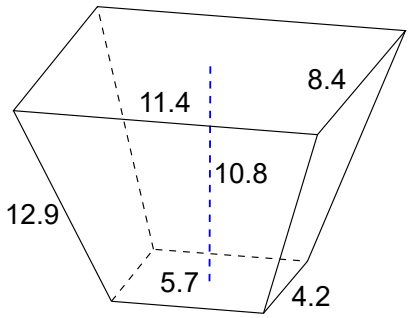
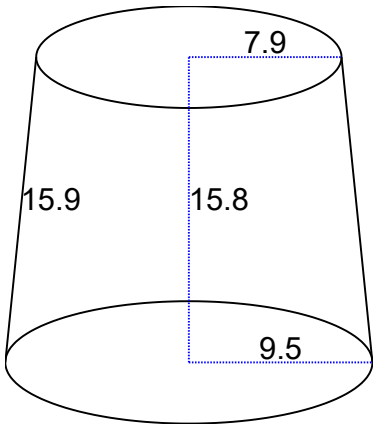


Image is not to scale. All measurements are in units of yd. Round according to the rules for measurements.

What is the volume of this frustum? _____ yd^3

What is the surface area of this frustum _____ yd^2

12.



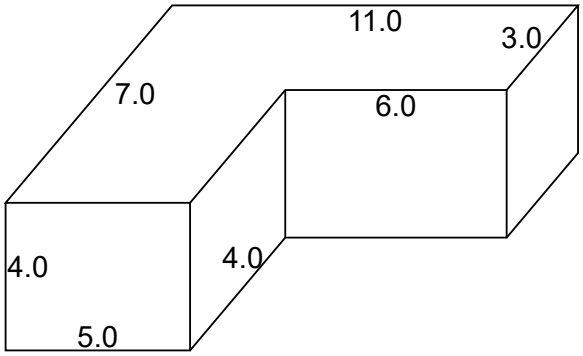
This is a frustum of a right cone with bottom radius 9.5, top radius 7.9 and height 15.8. Round results according to the rules for measurements.

What is the volume? _____

What is the surface area? _____

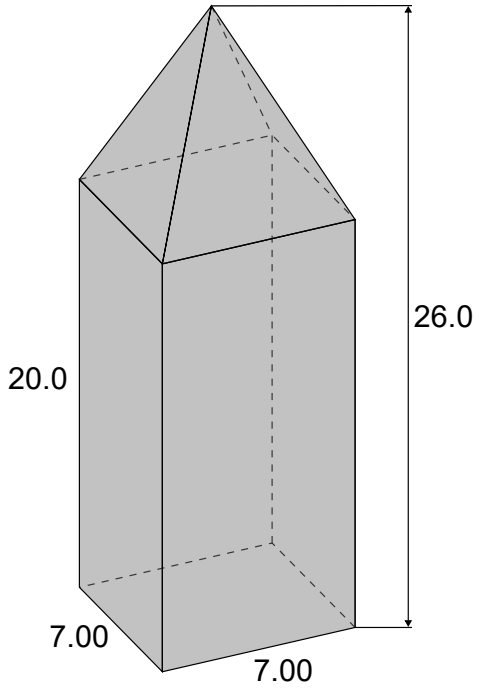
Compound Shapes. Identify the shapes from which each object is formed and use appropriate formulae to calculate the property requested.

13. Find the volume of the figure below. All measurements are in yards. Round accordingly.



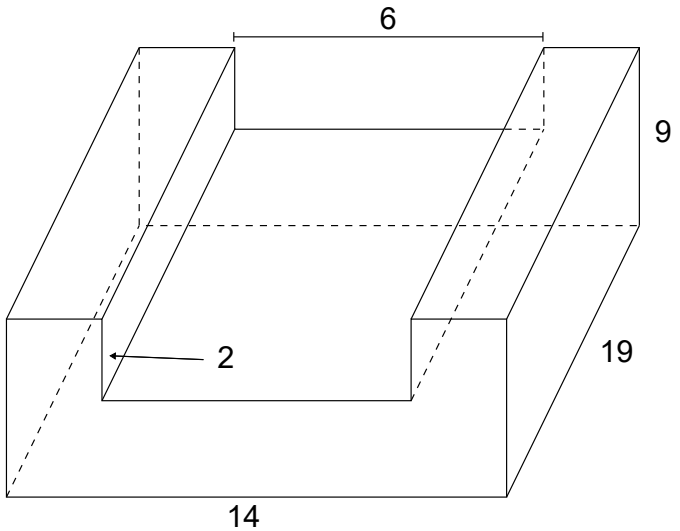
_____ yd^3

14. Find the volume of the composite figure below which consists of a square prims with a square pyramid on top. The sides of the square have length 7.00 cm. The height of the prism is 20.0 cm, and the total height is 26.0 cm.



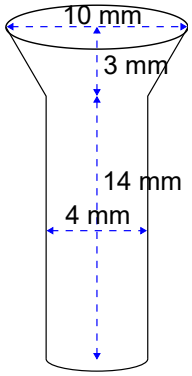
All measurements are in centimeters. Round according to rules for measurements.
Calculate the volume of just the square prism.
V_{prism} = _____ cm^3
Calculate the volume of just the pyramid.
V_{pyramid} = _____ cm^3
Calculate the composite volume.
$V_{\text{composite}}$ = _____ cm^3

15. Figure is not drawn to scale. All units are in centimeters. No rounding required.



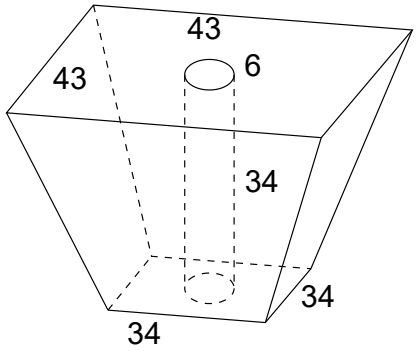
What is the volume of the box? _____ cubic centimeters

16.



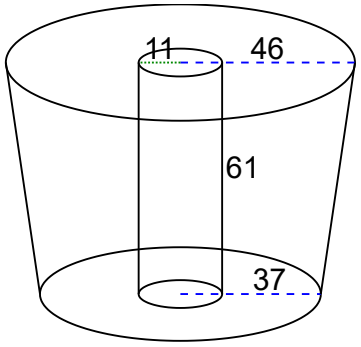
The image is not to scale. Round to units.
What is the volume of this object? _____ mm^3

17.



The image is not to scale. Round to units.
What is the volume of this shape with a hole in it? _____

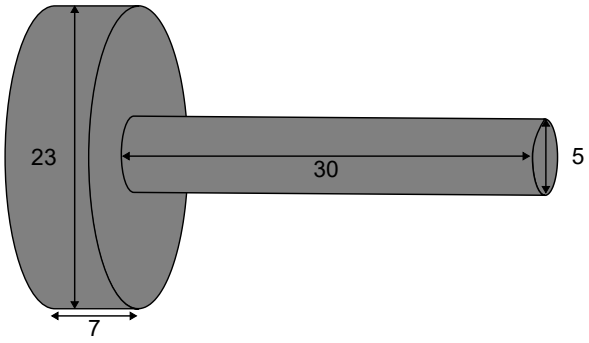
18.



This is a frustum of a right cone with bottom radius 37, top radius 46 and height 61 with a cylindrical hole of radius 11 in it. All measurements are in millimeters. Round results to the nearest unit.

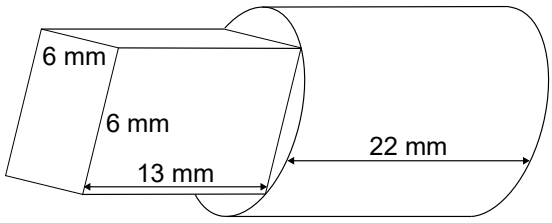
What is the volume? _____ mm^3

19.



The image is not to scale. Round to units.
What is the volume of this piston and rod? _____

20.



The image is not to scale. Round to units.
What is the volume of this object? _____

Geometry in Applications. Identify shapes and use appropriate formulae to calculate the results requested.

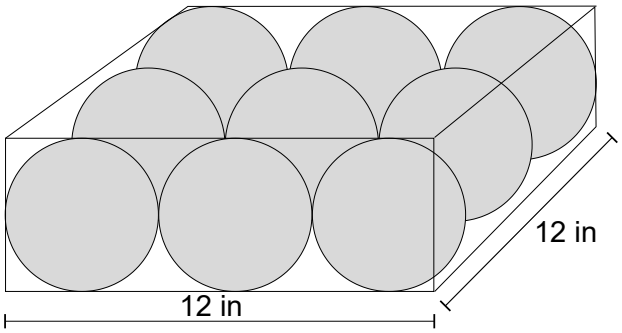
21. A cylindrical soup can has a diameter of 2.6 in and is 4.7 in tall. Find the volume of the can.



Round your answer using the rules for working with measurements.
_____ in^3

22. A box contains 9 identical glass spheres that are used to make snow globes. The spheres are tightly

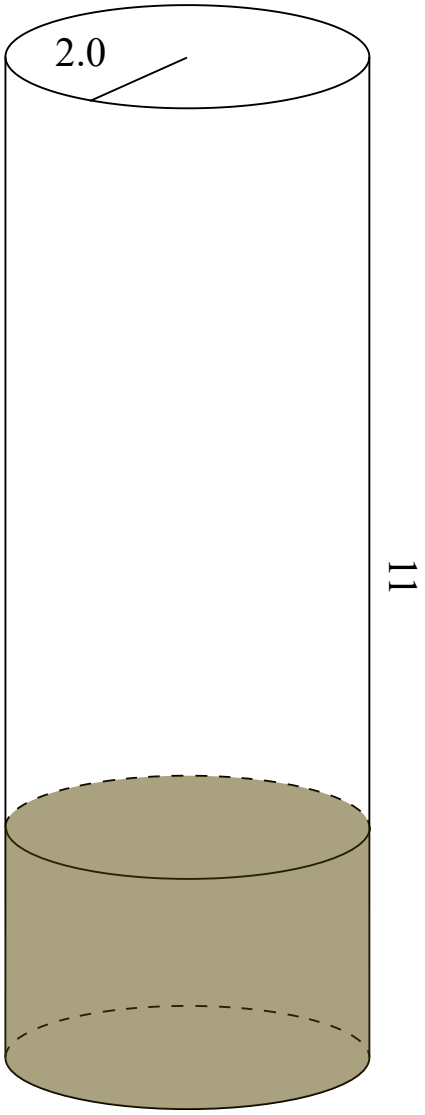
packed, as shown below.



What is the total volume, in cubic inches, of all 9 spheres? _____ cubic inches

If a styrofoam inset is made to fill the area between the box and the spheres, what is the volume of foam needed? _____

23. A grain silo is shown below. Measurements are in meters.



If 32 m^3 of grain is stored in the silo, what percent of the silo is full? _____

24. Leire is making candles in the shape of a cylinder. She needs to determine how many cubic centimeters of wax she needs.

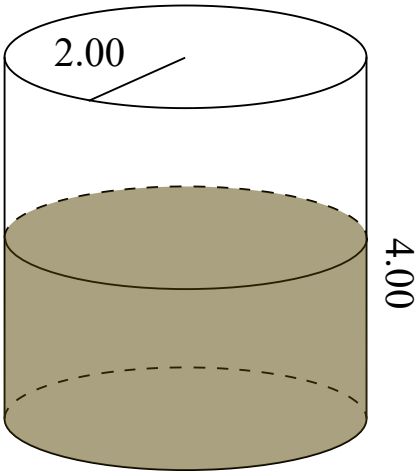


Suppose the radius of the candle is 2.0 cm and the height of the candle is 9.0 cm. Round using significant figures.

1. How much wax is needed for one candle? _____ cm^3
2. How much wax is needed for 20 candles? _____ cm^3
3. If wax is sold in cubic meters, how many *complete* candles can be made with 1 cubic meter of wax? _____ candles

Hint: 1 cubic m = 1,000,000 cm^3

25. A grain silo is shown below. The silo is filled halfway full with grain.
Measurements are in *meters*. Round your answer appropriately given the measurements.



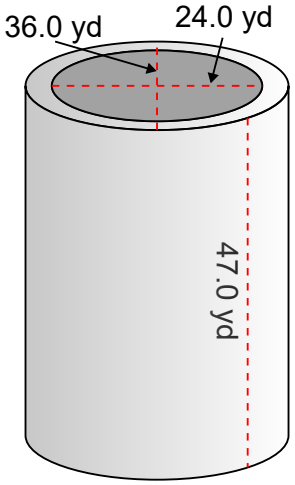
Determine the volume of grain in the silo.

_____

1. cubic meters
2. cubic yards
3. cubic feet
4. feet
5. meters
6. yards

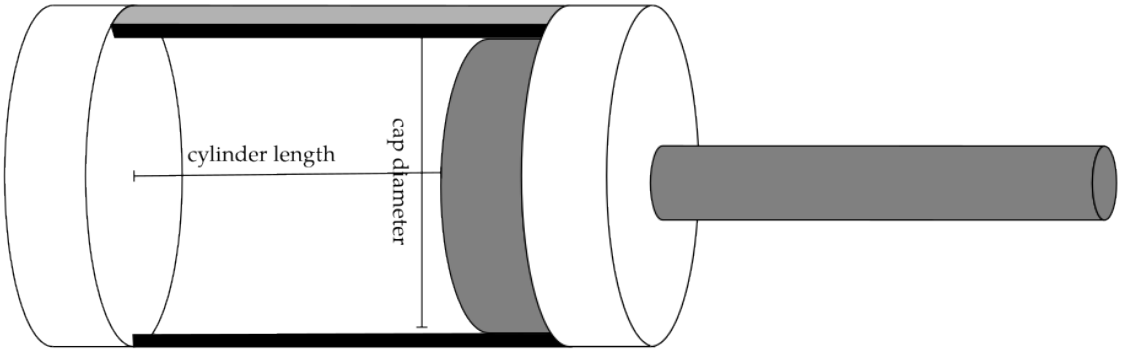
26. The diameter of the pipe shown is 36.0 yd. The inner diameter of the pipe is 24.0 yd. Note the image is not to scale. How much metal is in this pipe? Round your answer using the rules for

measurements.



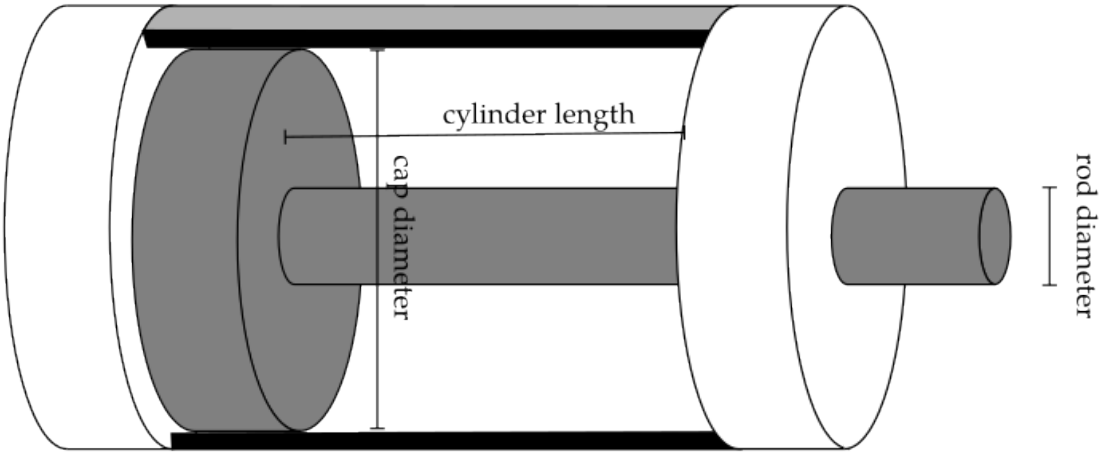
The pipe is made of ____ yd^3 of metal.

27.



The length of the cylinder is 8.9 feet. The cap diameter is 3.3 inches. The rod diameter is 1.6 inches. Round to the nearest cubic inch.

How much fluid is needed to fill the extended cylinder? _____ in^3



How much fluid is needed to fill the retracted cylinder? _____ in^3

The amount of fluid needed when the cylinder is extended is

1. less
2. the same

3. more

28. How many loads of gravel will be needed to cover 2.5 miles of roadbed, 33 feet wide, to a depth of 3.5 inches, if one truckload contains 7.9 yd^3 of gravel?

Remember that 1 mi = 5280 ft.

Find the volume of gravel needed, rounded to the nearest cubic foot.

_____ ft^3

Convert that volume to cubic yards.

_____ yd^3

Finally, how many truckloads? Think about appropriate rounding. Recall that one truckload contains 7.9 yd^3 of gravel.

_____ truckloads

29. Gravel is piled in the shape of a cone. The circumference of the base is 50.3 feet. The slant height is 17.0 feet. Find the volume of gravel.

Use the rules for working with measurements to give your answer to the appropriate level of accuracy or precision.

_____ ft^3

If the gravel weighs $3,200\frac{\text{lb}}{\text{yd}^3}$, how many 20-ton truckloads are needed to transport the gravel? Think about what the appropriate rounding would be for the number of truckloads.

_____ truckloads.

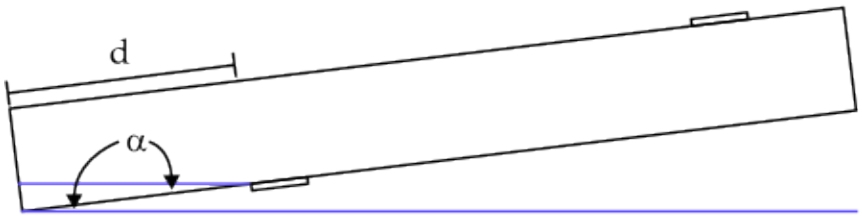
It may be useful to know the following:

3 feet = 1 yard

2000 lb = 1 ton

Quantitative Literacy. Use your understanding of geometry to address these questions.

30.



The image above illustrates a fuel tank tipped up. The port out of which fuel is pumped is closer to the left side of the tank than the right side. Due to the uphill angle there is an area (below the blue line) on the bottom, left that is below the port. Because it is below, that fuel cannot be used. The same problem would exist if the vehicle were headed downhill, but the volume of fuel would be different. Supposing the angle is the same (just down instead of up), there would be more unusable fuel when the vehicle is heading

1. up
2. down

4.3 Project: Unicorn Cake

Project 4.3.1 Building a Unicorn Cake. The purpose of this project is to use knowledge of geometric properties to solve a scaling problem.

You have been asked to bake the birthday cake of a little girl who is about to turn three. This little girl has been begging for weeks for a unicorn cake, and you decide to indulge her. When looking up suggestions online on how to bake this cake, you found that most everyone suggests baking the cake in smaller circular cake pans that have a diameter of 6 inches. That's the best way to get the height for the unicorn head. A

normal circular cake pan (and the only kind you have) has a diameter of 9 inches. You need to buy cake pans. As an experienced baker, you know that the recipe you are planning on using usually fills two regular (9-inch) cake pans, with a little room to spare. How many 6-inch pans do you need to buy? We're going to assume all cake pans are the same height.

In this project, we will figure out how many 6 inch circular cake pans we're going to need, assuming we're sticking with the recipe we planned.

Instead of a traditional assignment where the questions guide you from step to step, this project has you practice your math solving skills from start to finish. You will demonstrate your ability to answer the question without any leading prompts. This will likely take you about a page or less.

You will be graded on the following.

1. Is your thought process clear? Are the steps laid out in a logical manner? If you used any formulas, did you write them down and label them? Don't just race to the answer; set this up as if you are explaining it to someone else.
2. You need to show some calculations to justify your answer.
3. Some kind of written justification or explanation as to how you solved the problem. (Yes, use words!)
4. A clearly stated (and correct) answer.

(a) How many 6-inch cake pans will you need to use to bake a unicorn cake, assuming you're sticking with the original recipe?

(b) We "assumed all cake pans are the same height." Why is this important and how does it impact your solution?

4.4 Project: Watering Grass

Project 4.4.1 Watering Grass. Many common sprinklers water a circular area. Many lawns are much more rectangular. Placing the sprinklers to that they water enough of the lawn but not too much of the non-grass surfaces (such as driveways) requires thought. In this project we will consider three related questions.

(a) Our first attempt to water the entire lawn will space the sprinklers so that there is no overlap. This will result in not watering some portions of the lawn.

(i) The distance a sprinkler sprays can be set when it is installed. For this case to what distance (radius) should the sprinklers be set to produce this watering pattern?

The layout is illustrated in Figure 4.4.2.

(ii) What is the area that is not covered?

(iii) Sprinklers can be set to spray 360°, 180°, or 90°. For the layout in Figure 4.4.3, what distance (radius) should the sprinklers be set to produce this watering pattern?

(b) We next investigate how to modify the previous pattern to cover all the area. Use the lawn in Figure 4.4.4.

(i) Calculate the area covered by the circles and partial circles shown. The radius will be the same as in the first lawn.

(ii) If another sprinkler is placed at the center of the six unwatered areas (four pointed star-like shapes), to what radius should it be set to cover that whole area?

(iii) If we add these six sprinklers to the pattern, what is the total area of the lawn that is watered by more than one sprinkler?

(iv) What is the area of one of the overlap areas (the lens shaped regions).

(v) What percent is the overlap area with respect to the total area of the lawn?

(c) Now we consider an alternative overlapping pattern. Rather than add additional sprinklers to fill the gaps, we will move the sprinklers closer together so there are no gaps. This of course means we still need more sprinklers.

The pattern consists of four circles each touching a central point. From that point each circle's center is 45° from the next. The pattern is illustrated in Figure 4.4.5.

The main goal here is to calculate the area of the overlap.

(i) First, we calculate the area of the wedge shown in Figure 4.4.6. The interior angle of this wedge is 90°. As a result this wedge is what part of the whole? Your answer should be something like 20/360 or 1/7.

(ii) As a result of the previous answer, and supposing the radius is 15.9, what is the area of the wedge?

(iii) The overlap area is based on the area that is the difference between the wedge and the triangle. We know the area of the wedge, so lets calculate the area of the triangle. If the radius is 15.9, what is the area of this triangle? Remember this is a right triangle.

(iv) Calculate the area of the half lens (difference between the wedge and the triangle). The overlap area is twice this area (two lens halves). Calculate this overlap area.

(v) What is the total overlap area (all the lense and half lens shapes).

(vi) What percent of the total area is overlapped?

(d) Which sprinkler pattern had a lower percent of overlap?

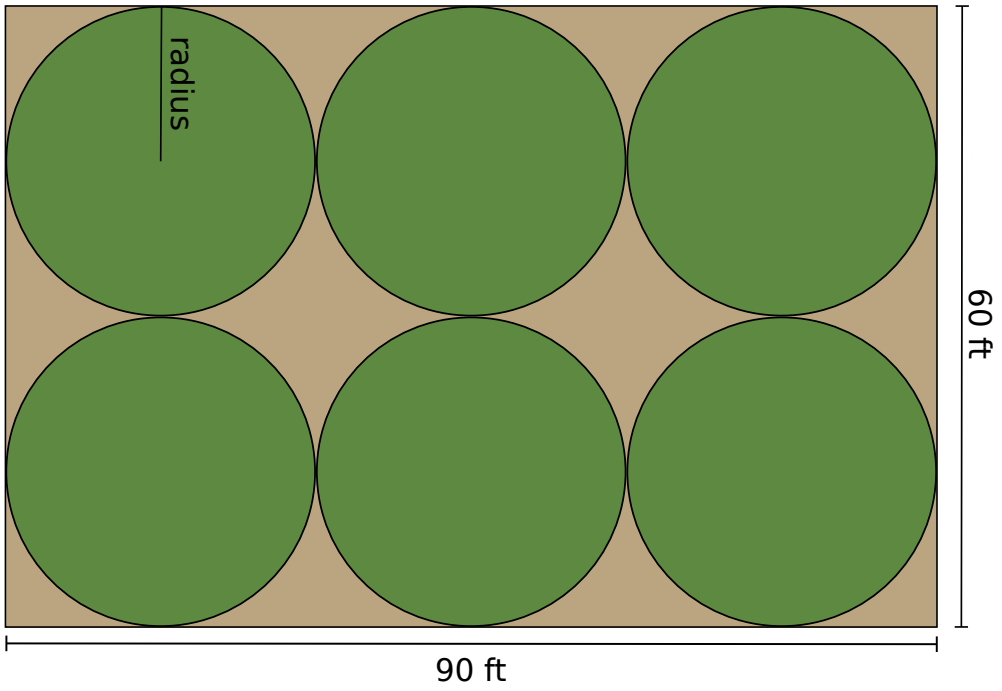


Figure 4.4.2 Sprinklers in Rectangular Grid

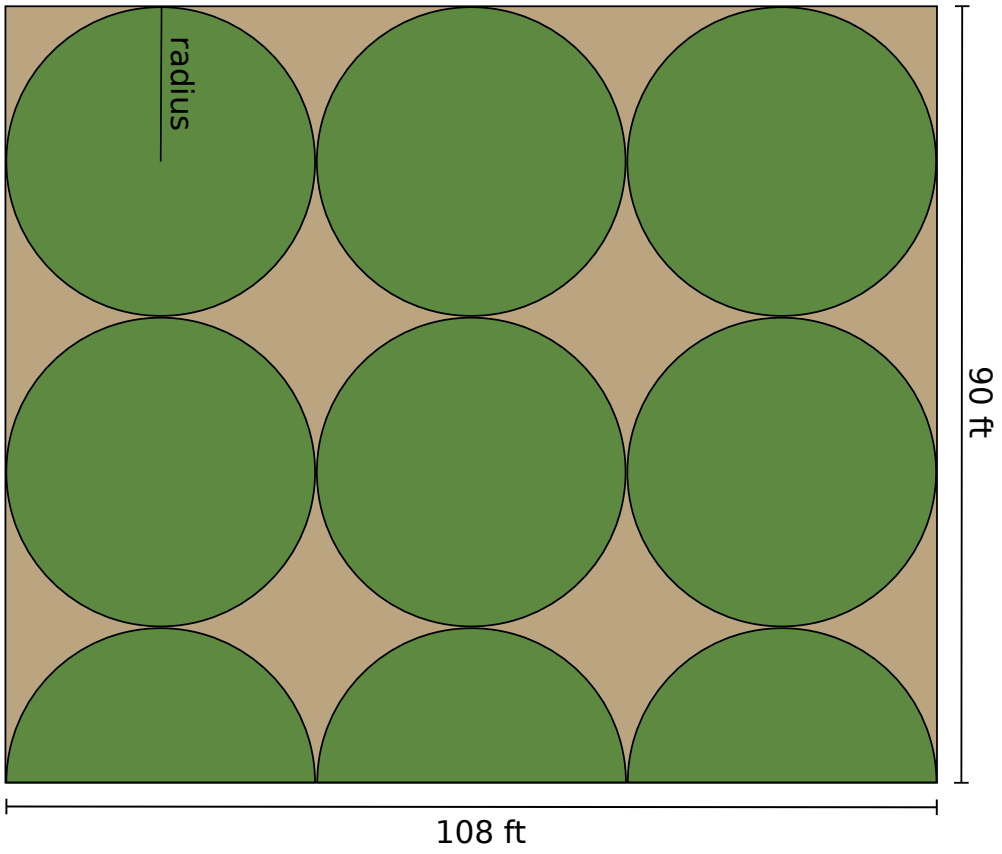


Figure 4.4.3 Sprinklers in Half Circle Mode

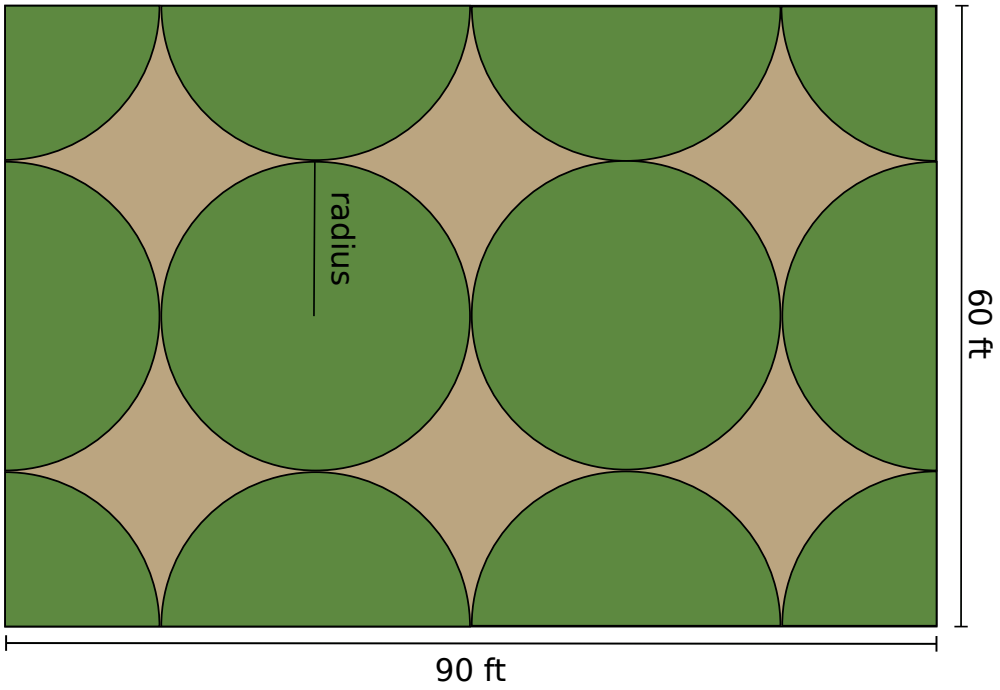


Figure 4.4.4 Sprinklers Arrangement to Extend

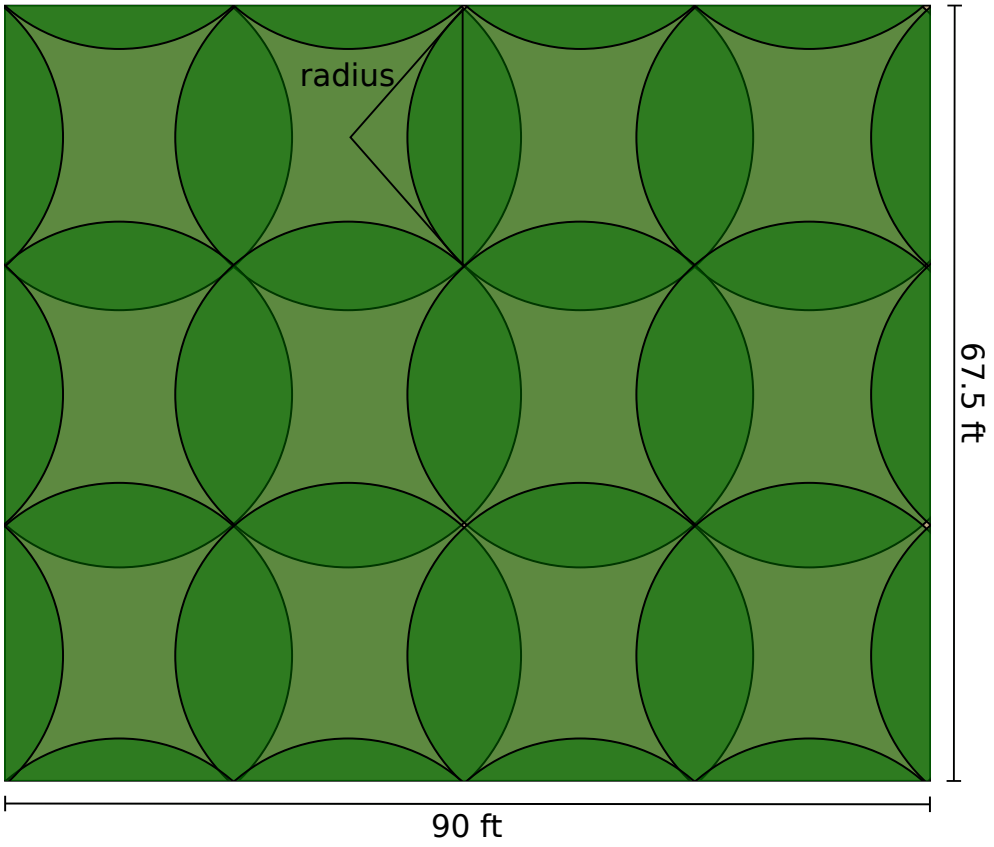


Figure 4.4.5 Sprinklers in Overlapping Rectangular Grid

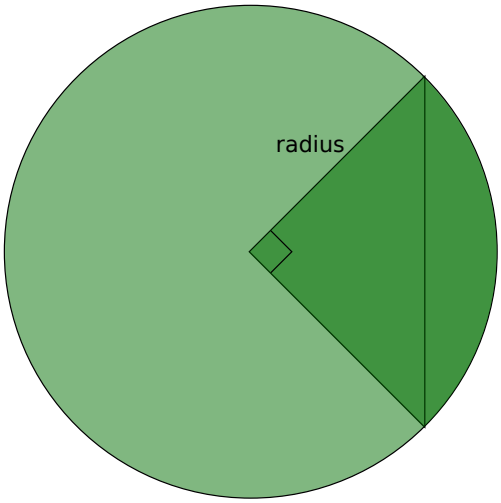
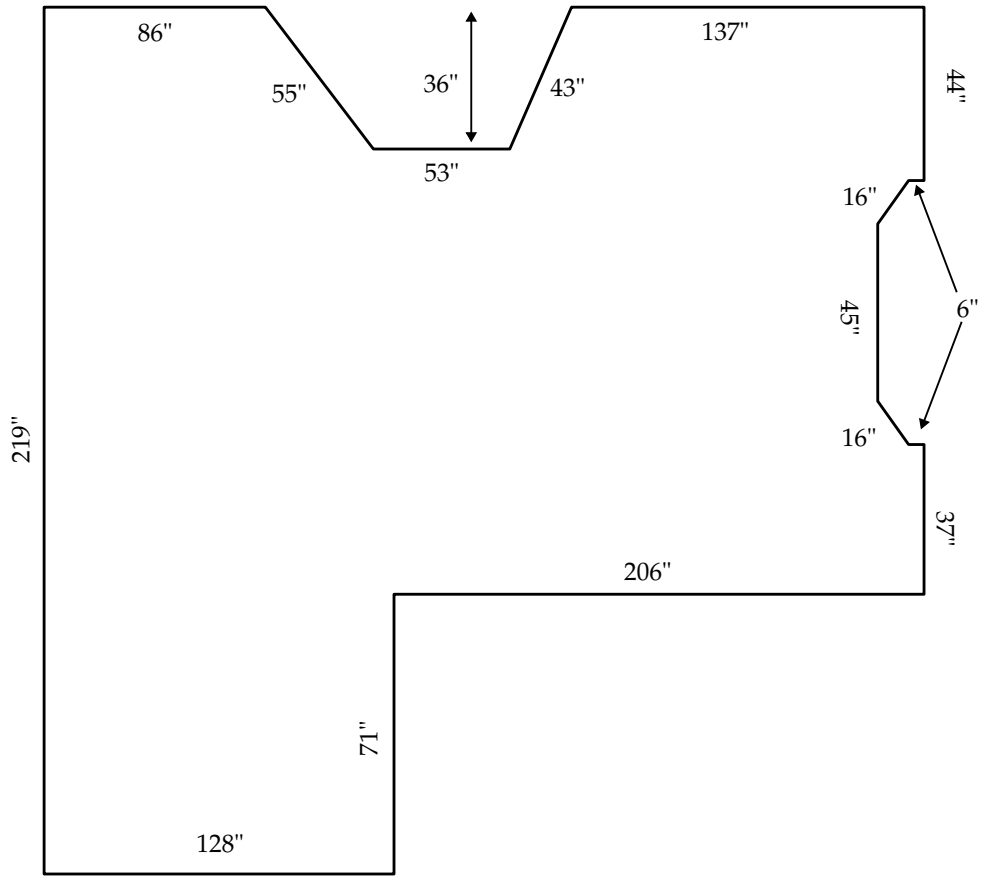


Figure 4.4.6 Identify Overlap Area

4.5 Project: Order Flooring

Project 4.5.1 Ordering Enough Flooring. The purpose of this project is to use knowledge of geometric properties to solve a purchasing problem. First, we must analyze a floor plan using our ability to break down a shape into easier parts. Second, we must use our knowledge of ratios and rounding to obtain a reasonable cost estimate.

Your carpet is worn out and you are looking to replace it with bamboo flooring. We need to know how much flooring to order and how much it will cost. This is dependent on the square footage of floor to be covered.



(a) The irregular, four sided area at the top of the floor plan is the entryway. You will be keeping that tile because that is easier to clean where dirty boots often go. You could order new tile, in a pattern to match the bamboo flooring, if it is not too expensive. What is the area in square inches of the entryway?

(b) The six sided area at the right of the floor plan is the wood stove. That is fire safe tile. You could also order new tile here if it is not too expensive. What is the area in square inches of the wood stove area?

(c) What is the total area that will be covered in the bamboo flooring?

(d) Because of mistakes and other limitations, it is recommended that we order 10% more than the calculated amount. How many square inches is this?

(e) Flooring is described by how many square feet it covers. How many square feet do we need?

(f) Each box of bamboo flooring is supposed to cover $27.4\,\text{ft}^2$. How many boxes should you order?

(g) A box of flooring costs \$96.16. How much will the bamboo flooring cost?

(h) A case of tile for the entryway contains 8 tiles. Each tile is 12×24 inches. How many cases would you need?

(i) A case of these tiles costs \$23.91. How much will this tile cost?

(j) A case of tile for the wood stove claims to cover $10\,\text{ft}^2$. Each tile is 14 inches square. How many tiles are in a box?

(k) A case of these tiles costs $157.90. How much will new tile for the wood stove area cost?

(l) You have $1,600.00 available for this project, and the bamboo flooring is the main goal. Which parts of the floor will you recover?

4.6 Project: Rounded Rectangle

Project 4.6.1 Rounded Rectangle. A common shape in graphic design is a rounded rectangle. This is a rectangle whose corners have been turned into quarter circles. These rounded corners can be small or large where this is measured by the size of the radius of the circle. In this project we determine the impact the radius of the corners has on the area of the shape. We also determine a maximum and minimum possible area.

(a) First, we practice calculating the area of a rounded rectangle and produce some data for later steps.

(i) Calculate the area of the rounded rectangle in Figure 4.6.2(a). There are multiple ways to calculate this area. You might calculate the area of the four circular corners and then the rest of the area.

(ii) Calculate the area of the rounded rectangle in Figure 4.6.2(b)

(iii) Which has more area, the rectangle with smaller corner radius (Figure 4.6.2(a)) or the rectangle with larger corner radius (Figure 4.6.2(b))?

(b) In this task we determine the first bound (maximum or minimum area) for a rounded rectangle.

(i) There is a maximum sized radius for the rounded corners. It results from two, adjacent corners running into each other. For a rectangle with width 70 and height 40, what is the maximum radius possible?

(ii) What is the area of the rounded rectangle with this radius?

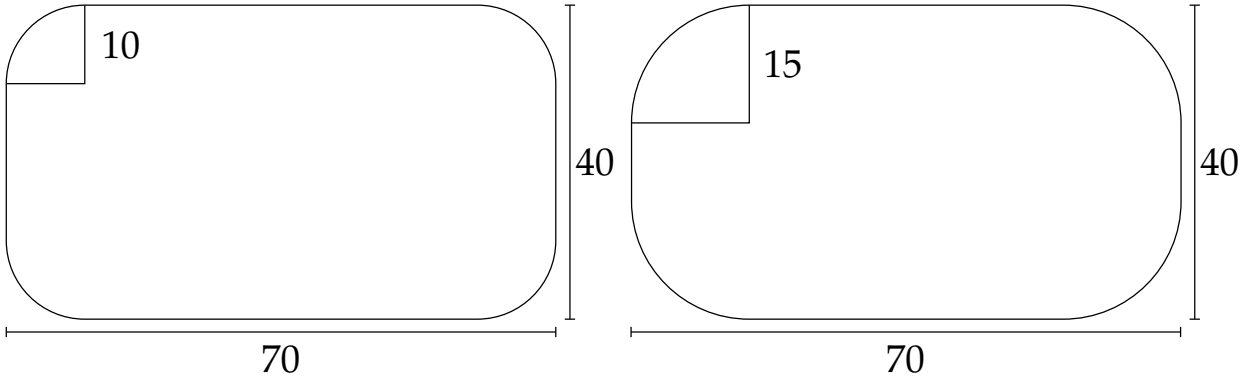
(iii) Is this the maximum or minimum area for a rounded rectangle with width 70 and height 40?

(c) In this task we determine the second bound (maximum or minimum area) for a rounded rectangle.

(i) As the radius of the corners becomes smaller, the area of the rounded rectangle becomes smaller or larger?

(ii) As the radius of the corners becomes smaller the area of a rectangle with width 70 and height 40 becomes closer to what area?

(iii) Is this the maximum or minimum area for a rounded rectangle with width 70 and height 40?



(a) Small Corners **(b)** Larger Corners

Figure 4.6.2 Rounded Rectangles

Chapter 5

Quadratics

5.1 Quadratics

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)
- Read and interpret models (critical thinking)

In Section 3.4 we learned to identify data that has a quadratic relation. This section presents algebraic notation for quadratics with emphasis on the forms we will use in this book. You should be able to identify a model as quadratic by looking at the equation.

One of comparing different rates is to enable us to enable us to provide more specific statements than "slow" or "fast." Quadratic is faster than linear. In the next chapter we will learn that exponential is faster than quadratic (or any polynomial). Frequently descriptions of rates in casual conversation and also in the media are lacking in detail or are even inaccurate. This is part of the concept "Identify rates as linear, quadratic, exponential, or other."

5.1.1 Algebraic Forms of Quadratics

Quadratic refers to any expression or equation that has a non-zero squared term. The three most common forms are below. All three rows have the same quadratic.

Table 5.1.1 Algebraic Notation for Quadratics

Example	Form	Description
$y = 5x^2 - 17x - 12$	$y = ax^2 + bx + c,\ a \neq 0$	Standard
$y = (5x + 3)(x - 4)$	$y = (a_1x - b_1)(a_2x - b_2),\ a_i \neq 0$	Factored
$y = 5\left(x - \frac{17}{10}\right)^2 - \frac{1489}{100}$	$y = a(x - h)^2 + k,\ a \neq 0$	Convenient

For this class the form we will see the most is

$$y = kx^2$$

where k has some meaning in each model.

When reading models quadratics may look a little different. Table 5.1.2 shows examples of equations that are quadratic and some that are not (but might look like it).

Table 5.1.2 Quadratic and Non-quadratic

Quadratic	Non-quadratic
$11x^2 + 32x - 3$	$5x + 3$
$2(x-3)^2 + 7$	$y = x^3 + 7x^2 - 5x + 3$
$y = 23 - 3x^2$	$y = \frac{17}{x^2}$
$x(6x - 5) = 21$	$x^2(x - 5) = 7$
$y = (x + 3)(x - 5)$	$y = 0x^2 + 3x + 2$

5.1.2 Quadratic Models

Now that we know how to recognize models (equations) as quadratic, we will present a few.

Model 5.1.3 Load Factor. *The maximum load factor on an aircraft is*

$$n_{max} = \left(\frac{v}{v_s}\right)^2$$

where

- *n is the load factor,*
- *v is the velocity, and*
- *v_s is the stall speed.*

Load factor is a measure of the force exerted on the aircraft by a maneuver. Maximum load factor is the greatest load factor an aircraft would experience if at the given speed, the aircraft executed a maximum performance maneuver.

Load factor is measured as a multiple of the force of gravity. Thus a load factor of 2 means the object is subject to a force twice as strong as earth's gravity. The expression "pulling g's" refers to experiencing a load factor greater than one.

Example 5.1.4 What is the maximum load factor if the stall speed is 54 kias, and the current speed is 95 kias? What if the current speed is 105 kias? How many g's does the maximum load factor increase between those two speeds? Recall kias is nm/hour. Round to one decimal place.

We use the formula

$$\begin{aligned} n_{max} &= \left(\frac{95}{54}\right)^2 \\ &\approx (1.759259259)^2 \\ &\approx 3.094993140 \\ &\approx 3.1. \end{aligned}$$

$$\begin{aligned} n_{max} &= \left(\frac{105}{54}\right)^2 \\ &\approx (1.9444444449)^2 \\ &\approx 3.780864196 \\ &\approx 3.8. \end{aligned}$$

The increase of 10 kias increased the g's by $3.8 - 3.1 = 0.7$. ▲

Model 5.1.5 Kinetic Energy. *The kinetic energy of an object in motion is given by*

$$E = \frac{1}{2}mv^2$$

where

- *E is the energy (in foot pounds or Joules)*
- *m is the mass (think weight) of the object (in pound mass or grams)*
- *v is the velocity of the object (in feet or meters per second).*

Example 5.1.6 What is the kinetic energy of a glider weighing 1323 lbs and flying at 34 kias, at 54 kias? How much did it increase? Use significant digits.

First, we need to convert the speeds to feet per second. Table 1.1.2 suggests we can multiply.

$$\frac{34 \text{ nm}}{\text{hr}} \cdot \frac{6076 \text{ ft}}{\text{nm}} \cdot \frac{\text{hr}}{3600 \text{ sec}} \approx 57.38444444.$$
$$\frac{54 \text{ nm}}{\text{hr}} \cdot \frac{6076 \text{ ft}}{\text{nm}} \cdot \frac{\text{hr}}{3600 \text{ sec}} \approx 91.14.$$

Now, we can calculate the kinetic energy.

$$\begin{aligned} E_{34} &= \frac{1}{2}(1323 \text{ lbs}) \left(\frac{57.38444444 \text{ ft}}{\text{sec}} \right)^2 \\ &\approx 2.178302608 \times 10^6 \\ &\approx 2.2 \times 10^6. \\ E_{54} &= \frac{1}{2}(1323 \text{ lbs}) \left(\frac{91.14 \text{ ft}}{\text{sec}} \right)^2 \\ &\approx 5.494749485 \times 10^6 \\ &\approx 5.5 \times 10^6. \end{aligned}$$

The increase was $5.5 \times 10^6 - 2.2 \times 10^3 = 3.3 \times 10^3$ foot pounds. ▲

Model 5.1.7 Potential Energy. *Potential energy of an object is given by*

$$E = mgh$$

where

- *m is the mass of the object (in pound mass or grams)*
- *g is the gravitational constant (in feet per seconds squared or meters per second squared)*
- *h is the height of the object (in feet or meters)*

Potential energy can be thought of as the energy that was expended to move the object higher that can be regained by letting gravity pull it back down. The height may be calculated as above ground level, or above sea level, or any other reference convenient for a question.

5.1.3 Inversely Quadratic Relations

Table 5.1.2 has $y = 17/x^2$ as an example that is not quadratic. Instead y varies inversely with the square of x. While not quadratic, as illustrated below by the tables of differences, they can be solved using the same techniques. Solving quadratics will be demonstrated in the next section.

Table 5.1.8 Quadratic Relation

x	$3x^2$	1st Difference	2nd Difference
1	3		
2	12	9	
3	27	15	6
4	48	21	6
5	75	27	6
6	108	33	6

Table 5.1.9 Inversely Quadratic Relation

x	$\frac{3}{x^2}$	1st Difference	2nd Difference
1	3		
2	3/4	-9/4	
3	1/3	-5/12	11/6
4	3/16	-7/48	13/48
5	3/25	-27/400	47/600
6	1/12	-11/300	37/1200

In Table 5.1.8 the second differences are all 6, so that is quadratic. In contrast in Table 5.1.9 the second differences are all different.

Model 5.1.10 Gravitational Attraction. *Two objects exert a gravitational pull on each other related by their masses and the distance between them. The relationship is*

$$F = G \cdot \frac{m_1 m_2}{r^2}$$

where

- F *is the resulting gravitational force in Newtons*
- G *is the gravitational constant*
- m_1, m_2 *are the masses of the two objects in kilograms*
- r *is the distance between the two objects in meters*

The gravitational constant is $G = 6.6743 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$. *The constant has not been determined to any greater precision.*

Example 5.1.11 What force does the earth exert on the moon? The mass of the earth is approximately 5.97219×10^{24} kg, and the mass of the moon is approximately 7.34767×10^{22} kg. The average distance between the earth and moon is 382,500 km.

We need to convert the kilometers to meters so units match (gravitational constant). Because it is kilo the conversion is $382,500 \text{ km} \cdot \frac{1000 \text{ m}}{\text{km}} = 382,500,000 \text{ m}$. Using the model

$$\begin{aligned} F &= (6.6743 \times 10^{-11}) \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \left(\frac{(5.97219 \times 10^{24} \text{ kg})(7.34767 \times 10^{22} \text{ kg})}{(382500000 \text{ m})^2} \right) \\ &\approx 2.001824977 \times 10^{20} \frac{\text{m kg}}{\text{s}^2} \\ &\approx 2.002 \times 10^{20} \text{N} \end{aligned}$$

▲

5.1.4 Exercises

1. Recall that the lift equation is $L = \frac{1}{2}\rho s C_L v^2$. Because these are measurements round using significant digits.

Suppose $\rho = 0.001905$, $s = 279$, and $C_L = 1.26$. How much does the lift increase if velocity is increased from 36.7 nm/hr to 49.8 nm/hr? _____

2. Two aircraft are flying at 70 nm/hr. The Diamond Eclipse has a stall speed of $v_s = 34$ nm/hr, and the Diamond DA40 NG has a stall speed of $v_s = 62$ nm/hr.

The Diamond Eclipse has a maximum load factor that is _____ G's

1. lower
2. higher

Round to 1 decimal places.

3. Consider a Toyota Prius weighing 3422 lbs.

How much does the kinetic energy increase from 29 to 39 mph? _____

How much does the kinetic energy increase from 39 to 49 mph? _____

In these units kinetic energy is $\frac{1}{2}\left(\frac{m}{g}\right)v^2$. g is the gravitational constant: $g = \frac{32\text{ft}}{\text{sec}}$. The energy will be in foot pounds which results from mass in pound mass and velocity in feet per second. Round using the rules of measurements.

4. Consider a communications satellite with mass of 2765 kg.

What is the gravitational force at an altitude of $4,425$ km? _____

What is the gravitational force at an altitude of $35,785$ km? _____

Did the force decrease/increase at higher altitude?

1. decrease
2. increase

The model is $F = \left(6.6743 \times 10^{-11}\right)\frac{\left(5.97219 \times 10^{24}\right)m}{r^2}$ where m is the mass of the satellite and r is the altitude of the satellite in meters. Round using the rules of measurements.

5. When sound travels from one material into another, some of the sound (energy) is reflected rather than transferring into the new material. The percent reflected is given by $R = \frac{\left(Z_2 - Z_1\right)^2}{\left(Z_2 + Z_1\right)^2} \times 100$ where Z_1 and Z_2 are the acoustic impedance values for the two substances. Given that $Z_1 = 36777900$ for brass and $Z_2 = 33152200$ for cast iron, what is the reflection percent? _____ %

Round to 2 decimal places.

6. The length of the near field of an ultrasonic tester is given by $L = \frac{D^2}{4\lambda}$ where D is the diameter of the transducer (depends on crystal size) and λ is the wavelength. If $D = 0.017$ and the wavelength is $\lambda = 975$ in 4340 steel, what is the length of the near field? _____

5.2 Solving Quadratics

This section addresses the following topics.

- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)

This section presents how to solve equations that contain quadratics. For this text we present two methods of solving quadratics. Both should be memorized and practiced.

5.2.1 Solving Quadratics with Inversion

The first method is for very simple quadratics of the form $ax^2 + c = 0$. Notice that there is no linear term in x. This is what makes solving these simpler. The majority of quadratic models in this text can be solved with this technique.

Example 5.2.1 Identify all solutions to $10 - 21x^2 = 0$. Round solutions to 2 decimal places.

We can solve this by undoing each operation.

$$\begin{aligned}
10 - 21x^2 &= 0. && \text{Subtract to isolate} x\\
-21x^2 &= -10. && \text{Divide to isolate } x\\
x^2 &= \frac{-10}{-21}.\\
x^2 &= \frac{10}{21}. && \text{Square root to undo square.}\\
\sqrt{x^2} &= \sqrt{10/21}.\\
x &= \pm\sqrt{10/21}.\\
x &\approx \pm 0.6900655593.\\
x &\approx \pm 0.69.
\end{aligned}$$

Notice that we end up with two results. The $\pm$ results from squaring eliminating a negative. That is, $(2)^2 = 4$ and $(-2)^2 = 4$. So $\sqrt{4}$ could be either 2 or -2. ▲

Example 5.2.2 Identify all solutions to $7(x-3)^2 - 4 = 0$. Round solutions to 2 decimal places.

Solution. We can solve this by undoing each operation.

$$\begin{aligned}
7(x-3)^2 - 4 &= 0.\\
7(x-3)^2 &= 4.\\
(x-3)^2 &= \frac{4}{7}.\\
\sqrt{(x-3)^2} &= \sqrt{\frac{4}{7}}.\\
x - 3 &= \pm\sqrt{\frac{4}{7}}.\\
x &= \pm\sqrt{\frac{4}{7}} + 3.\\
x &\approx 3.755928946, 2.244071054.\\
x &\approx 3.76, 2.24.
\end{aligned}$$

▲

Checkpoint 5.2.3 What are the solutions to $9(x+7)^2 - 18 = 0$? _____

Round solutions to two decimal places.

Significant Digits beyond Arithmetic. There are significant digits rules for addition/subtraction and multiplication/division. We can handle signficant digits with exponents, such as $(0.0230)^2$, because this can be interpreted as multiplication. In the case of $(0.023\underline{0})^2 = (0.023\underline{0})(0.023\underline{0}) = 0.00052\underline{9}$.

For square roots we rely on mathematics we do not need to present here that indicates we can calculate the square root so that the number of significant digits is maintained. For example $\sqrt{1230} \approx= 35.07135583 \approx 35.1$

Example 5.2.4 Recall the lift equation in Model 1.3.3. If $\rho = 0.002309$ slugs/ft^3, $S = 174.0$ ft^2, and $C_L = 0.5001$, what velocity in nautical miles per hour is needed to produce $L = 3500.$ lbs?

We start by filling in the information we know in the equation.

$$\begin{aligned} 3500. &= \frac{1}{2}(0.002309)(174.0)(0.5001)v^2. \\ 3500. &\approx 0.1004615883v^2. \\ \frac{3500.}{0.1004615883} &\approx v^2. \\ 34839.18639 &\approx v^2. \\ \sqrt{34839.18639} &\approx \sqrt{v^2}. \\ 186.6525821 &\approx v. \\ 186.7 &\approx v. \end{aligned}$$

We do not consider the negative square root, because a negative velocity (flying backwards) does not work. Remember to consider reality constraints.

Note the units for velocity are feet per second. Now we need to convert units (like Example 1.1.20).

$$\frac{186.6525821 \text{ ft}}{\text{s}} \cdot \frac{\text{mi}}{6076 \text{ ft}} \cdot \frac{3600 \text{ s}}{\text{hr}} \approx 110.5907333 \frac{\text{mi}}{\text{hr}} \approx 110.6 \frac{\text{mi}}{\text{hr}}.$$

▲

Example 5.2.5 Consider Model 5.1.3 What velocity will produce a maximum load factor of $n_{max} = 2$ if the stall speed is $v_s = 54$?

Note that 2 is an exact number, and 54 is a measurement.

Solution.

$$\begin{aligned} 2 &= \left(\frac{v}{v_s}\right)^2. \\ 2 &= \left(\frac{v}{54}\right)^2. \\ \sqrt{2} &= \sqrt{\left(\frac{v}{54}\right)^2}. \\ \sqrt{2} &= \frac{v}{54}. \\ 54\sqrt{2} &= 54\frac{v}{54}. \\ 54\sqrt{2} &= v. \\ 76.36753235 &\approx v. \\ 76 &\approx v. \end{aligned}$$

The aircraft would have to be travelling at 76 nm/hour to be able to experience 2 G's in a maximum performance maneuver.

Again reality constraints eliminate the possibility of the negative square root. Negative velocity (backwards motion) does not produce flight. ▲

5.2.2 Solving Quadratics with the Formula

When the quadratic has more than a square term, e.g., $11x^2 + 32x - 3 = 0$ we cannot solve by undoing each operation. In this text we will solve all of these quadratics using the quadratic formula. For $ax^2 + bx + c = 0$

the solutions are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 5.2.6 Identify all solutions to $11x^2 + 32x - 3 = 0$.

We note that $a = 11$, $b = 32$, $c = -3$.

$$\begin{aligned} x &= \frac{-32 \pm \sqrt{(32)^2 - 4(11)(-3)}}{2(11)} \\ &= \frac{-32 \pm \sqrt{1024 + 132}}{22} \\ &= \frac{-32 \pm \sqrt{1156}}{22} \\ &= \frac{-32 \pm 34}{22} \\ &= \frac{1}{11}, -3 \end{aligned}$$

▲

Checkpoint 5.2.7 What are the solutions to $-3x^2 + 7x + 3 = 0$? _____

Round results to 2 decimal places.

One place quadratics of this sort shows up is equilibrium states for chemical reactions. Search for ICE (initial, change, equilibrium) problems for a full explanation. The following example illustrates how to generate the quadratic and use the formula to solve for the desired result.

Example 5.2.8 Chemistry ICE. Consider the following chemical reaction.

$$Br_2 + Cl_2 \rightleftharpoons 2BrCl.$$

If the amount of bromine gas is 0.60 moles and the amount of chlorine gas is 0.60 moles, and the equilibrium constant $K_c = 7.0$ what are the amounts of each gas at equilibrium?

The ICE (initial, change, equilibrium) table for this is below.

Table 5.2.9 ICE Table

	Br_2	Cl_2	$2BrCl$
I	0.6	0.6	0
C	$-x$	$-x$	$+2x$
E	$0.6 - x$	$0.6 - x$	$0 + 2x$

The initial conditions (first row) will always be given (we know with what we start). The second row uses the x to indicate that we do not know how much will be converted by the reaction. It is $+2x$ for the result because when one molecule each of Br_2 and Cl_2 break apart there are 2 atoms each of Br and Cl resulting in two molecules of BrCl. This all comes from the chemical reaction.

The equilibrium equation then is

$$K_c = 7.0 = \frac{[BrCl]^2}{[Br_2][Cl_2]} = \frac{(2x)^2}{(0.6 - x)(0.6 - x)}.$$

This is always the ratio of equilibrium amounts for the outputs divided by equilibrium amounts for the inputs. It will be provided in this text.

Finally, we can solve this equation to determine x and the resulting amounts of each gas.

$$\begin{aligned} 7.0 &= \frac{(2x)^2}{(0.6 - x)(0.6 - x)} \\ (2x)^2 &= 7.0(0.6 - x)(0.6 - x) \\ 4x^2 &= 2.52 - 8.4x + 7.0x^2 \end{aligned}$$

$$0 = 2.\underline{5}2 - 8.4x + 3.0x^2$$

Next, we can solve for x using the quadratic formula.

$$\begin{aligned} x =& \frac{-(-8.4) \pm \sqrt{(-8.4)^2 - 4(3.0)(2.\underline{5}2)}}{2(3.0)} \\ =& \frac{8.4 \pm \sqrt{7\underline{0}.56 - 3\underline{0}.24}}{6.0} \\ =& \frac{8.4 \pm \sqrt{4\underline{0}.\overline{32}}}{6.0} \\ =& \frac{8.4 \pm 6.\underline{3}498}{6.0} \\ =& 0.3\underline{4}17, 2.\underline{4}583 \end{aligned}$$

Because the initial amounts are only 0.6, only the first number is a possible result. 0.34 moles of each gas will be changed resulting in $2(0.34) = 0.68$ moles of BrCl. ▲

5.2.3 Exercises

Solving Quadratic Equations. Use these contextless problems to practice using the quadratic formula.

1. Solve by the quadratic formula. List the solutions, separated by commas.
$-20x^2 - 21x - 4 = 0$
$x =$ _____

2. Solve using the quadratic formula. List your answers, separated by commas.
$2x^2 - 6 = 0$
_____

3. Solve using the quadratic formula. List your answers, separated by commas.
$2x^2 - 6 = 0$
_____

4. Solve by the quadratic formula. List the solutions, separated by commas.
$-20x^2 - 21x - 4 = 0$
$x =$ _____

Quadratic Applications. Answer these questions which involve a quadratic model. Select rounding that is appropriate or as directed.

5. A rocket launch occurs at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h = -4.9t^2 + 43t + 164$.
Assuming that the rocket will splash down into the ocean, at what time does splashdown occur? _____ seconds.
At what time will the rocket become level with the take-off location? _____ seconds

6. An aircraft in flight has total energy given by $E_T = \frac{1}{2}mv^2 + mgh$ where
m is the mass of the aircraft in pounds
v is the velocity in feet/second
$g = 32.1740$ is earth's standard gravity
h is the height above sea level in feet
Suppose a glider with mass 1111 lbs and initial velocity 92 ft/s is at an altitude of $9,600.$ ft MSL. If the glider forces a descent to $9,409$ ft MSL, what must the new velocity be? _____ ft/s
Round according to the rules for measurements.
Hint: The total energy will be the same before and after the altitude change.

7. An aircraft in flight has total energy given by $E_T = \frac{1}{2}mv^2 + mgh$ where

m is the mass of the aircraft in pounds
v is the velocity in feet/second
$g = 32.1740$ is earth's standard gravity
h is the height above sea level in feet

Suppose a glider with mass 1275 lbs and initial velocity 94.0 ft/s is at an altitude of 1000. ft MSL. If the pilot wants to increase the velocity to 144.0 ft/s to what altitude must it descend? _____ ft MSL

Round according to the rules for measurements.

Hint: The total energy will be the same before and after the altitude change.

8. Load factor refers to the amount of force imposed on an aircraft's body and contents (e.g., pilot). It is typically measured in g-forces. The maximum load factor is calculated by

$n_{\max} = \left(\frac{V}{V_s}\right)^2$ where

$n_{\max}$ is the maximum load factor
V_s is the stall speed, and
V is the current speed.

The stall speed (V_s) for a Diamond Eclipse is 34 kias (nm/hr). Calculate the V speed for a maximum load factor of 2. $V =$ ____

Round to the nearest unit.

9. The lift equation is $L = \frac{1}{2}\rho S C_L v^2$ where ρ is the density of air, S is the surface area of the airfoil (wing), C_L is called the coefficient of lift, and L is the lift (measured in pounds).

Calculate the velocity necessary to produce $L = 3300$ lbs of lift if $\rho = 0.001988$, $S = 174$, and $C_L = 4.5$.

$v =$ _____ ft/s

Round according to the rules for measurements.

10. The length of the near field of an ultrasonic tester is given by $L = \frac{D^2}{4\lambda}$ where D is the diameter of the transducer (depends on crystal size) and λ is the wavelength.

If we want the near field length to be $L = 1.356 \cdot 10^{-8}$ meters and we want to use the frequency with wavelength $\lambda = 1295$ cycles per meter in tungsten, what diameter in centimeters do we need? _____

Round according to the rules for measurements.

11. An approximation of the speed at which a vehicle will hydroplane on water (complete loss of traction) is

$V = 10.2\sqrt{p}$
where
V is the velocity in miles per hour
p is the tire pressure in psi

You plan to travel no faster than 65.0 miles per hour. To be somewhat safe from hydroplaning, what pressure should your tires be over? ____ psi

These are science measurements. Round appropriately.

12. Given the equilibrium expression $K = \frac{[CO_2][H_2]}{[CO][H_2O]}$, where $K = 3.10$ and the equilibrium concentrations below, calculate the amount changed x.

$[CO_2] = 1.00 + x$
$[H_2] = 1.00 + x$
$[CO] = 1.00 - x$
and
$[H_2O] = 1.00 - x$

This is a chemistry question, round using significant digits.

$x =$ _____
Hint: Start by plugging in all values and variables into the equilibrium expression.

Quantitative Literacy. Apply your knowledge of quadratics and solving them to answer these questions.

13. An aircraft in flight has total energy given by $E_T = \frac{1}{2}mv^2 + mgh$ where
m is the mass of the aircraft in pounds
v is the velocity in feet/second
$g = 32.1740$ is earth's standard gravity
h is the height above sea level in feet
We want to know how the velocity changes when the height changes.
As altitude increase the velocity will

1. increase
2. decrease

If needed try calculating velocities for two altitudes examples. For a glider with mass 1226 lbs and total energy 270413670.8, determine velocities at altitudes of 6700 and 6800 ft MSL.

5.3 Roots

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)

This section is for those who are curious for more details about roots, like the square root. These extensions do not show up in any models in this text. The rate question is an important general concept however.

5.3.1 Definition of Roots

As implied by their use in solving equations with a square term, square roots are a sort of opposite to squares. For example $3^2 = 9$ means that 9 is the result of multiplying 3 by itself. $\sqrt{9} = 3$ means that 3 is a number that multiplied by itself is 9. This means in general

$$\sqrt{n^2} = n.$$

There is one detail that we will not use, but should be acknowledged. Consider that $(3)^2 = 9$ and $(-3)^2 = 9$. Thus $\sqrt{9}$ might be considered to have two solutions. We saw this in Subsection 5.2.1. However when solving using inversion as in Subsection 5.2.1 reality constraints often meant the negative root was not a solution to the application even if it fits the equation.

5.3.2 Generalized Roots

Just as we can multiply a number by itself (e.g., $3^2 = 3 \cdot 3 = 9$) we can multiply a number by itself more than once (e.g., $3^3 = 3 \cdot 3 \cdot 3 = 27$). In general

$$3^m = \overbrace{3 \cdot 3 \cdot \ldots 3}^{m}.$$

Checkpoint 5.3.1 Evaluate each of these by multiplying enough times.

$4^5 =$ _____

$6^3 =$ _____

$2.8^4 =$ _____

As noted above square roots perform the reverse action of squaring. To solve problems involving other powers, there are matching roots. These are denoted with a small number to show which root. For example this is a third root:

$$\sqrt[3]{8} = 2.$$

This root means that because $2^3 = 8$, $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$. In general

$$\sqrt[n]{x^n} = x.$$

There are restrictions (n is an integer; if n is even, x must be non-negative). For models in this text none of these play a role.

Just as with square roots we can use a device to calculate the values. The device may have a key like $\sqrt[m]{x}$.

Another notation for roots is a type of exponent. For example,

$$\sqrt{9} = 9^{1/2} = 3.$$

Likewise,

$$\sqrt[3]{8} = 8^{1/3} = 2.$$

This notation can be used when solving problems with powers.

Checkpoint 5.3.2 Evaluate each of these. You may need a computational device.

$\sqrt[3]{343} =$ _____

$\sqrt[3]{512} =$ _____

$\sqrt[5]{7776} =$ _____

$\sqrt[4]{19} =$ _____

5.3.3 Rates of Roots

This section illustrates where roots fall with respect to the question, "How fast is it growing?" It ends by addressing an important general question: if growth is slowing is it stopping?

Figure 5.3.3 shows a linear equation and a square root equation. The square root eventually grows much more slowly than the linear grows. Because a linear grows more slowly than a quadratic or exponential, the root also grows more slowly than quadratics and exponentials.

The first important idea here is that the square root grows more quickly than the linear at first, but then slows down and is passed up. When we are comparing rates, we typically look at what happens eventually. It does not matter so much what happens at first. For example for salaries, exponential growth might be slower for the first few years, but in our careers it will be substantially more.

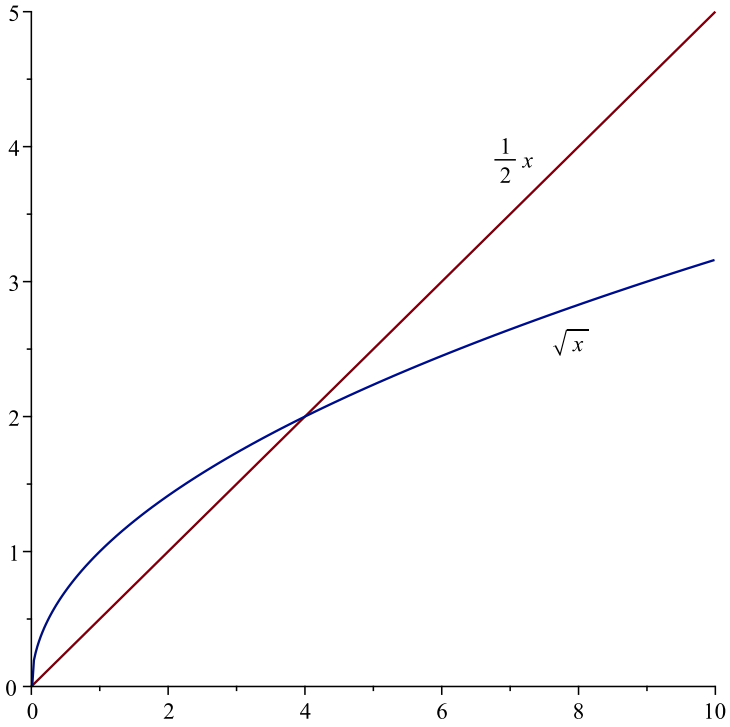


Figure 5.3.3 Rate Comparison: Linear and Square Root

With rates it is important to test if a slow growth rate is leveling off or not. By level off we mean that even though it is constantly growing it never passes some fixed amount. For example, the numbers 0.5, 0.75, 0.875, 0.9375, 0.96875 is growing, but never passes 1.0. We will show that a square root (and other roots) do not level off.

Consider Table 5.3.4. Clearly this grows larger than 1, because we have a 1 in the first entry. It also grows larger than 2, because we can obtain 2 from $n = 4$. In general if we ask, can the square root grow beyond n, the answer is: consider that $\sqrt{(n^2)} = n$, so if we choose $n^2 + 1$ that will be larger than n. While slow, the square root continues to grow without bound (infinite growth).

Table 5.3.4 Rate of a Square Root

n	$\sqrt{n}$
$1^2 = 1$	1
$2^2 = 4$	2
$3^2 = 9$	3
$10^2 = 100$	10
n^2	n

5.4 Graphs of Quadratics

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

This section will introduce the graph (shape) of quadratic equations, illustrate all the variations that are possible, and demonstrate how these can be used to answer questions.

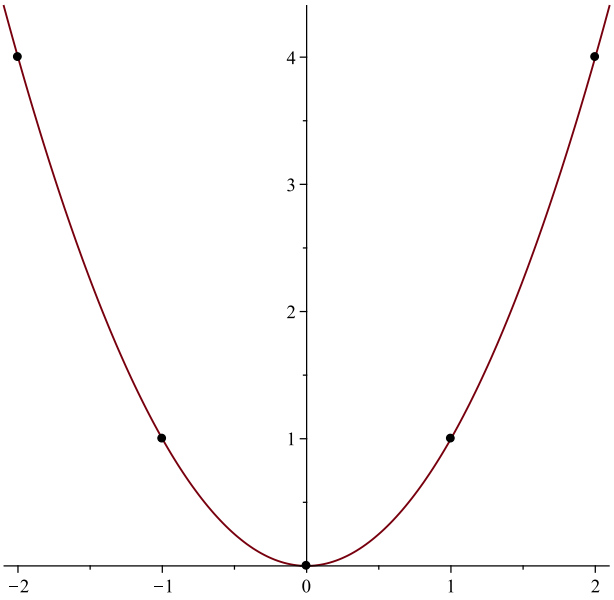
5.4.1 Properties of Quadratics

First we will graph the most basic quadratic equation $y = x^2$. This will show us the basic shape and properties. Later examples will show us the variations and how to control them.

Example 5.4.1 To discover the shape we will graph following the steps shown first in Subsection 3.2.3. First we complete this table of values. We must choose the x values randomly for now.

x	x^2
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$

Next we graph these points and sketch the graph through them.



▲

The graph of a quadratic is known as a **parabola**. Notice that there is a single point at the bottom from which the parabola grows upward to the left and upward to the right. This is known as the **vertex**. It is at (0,0) for this parabola.

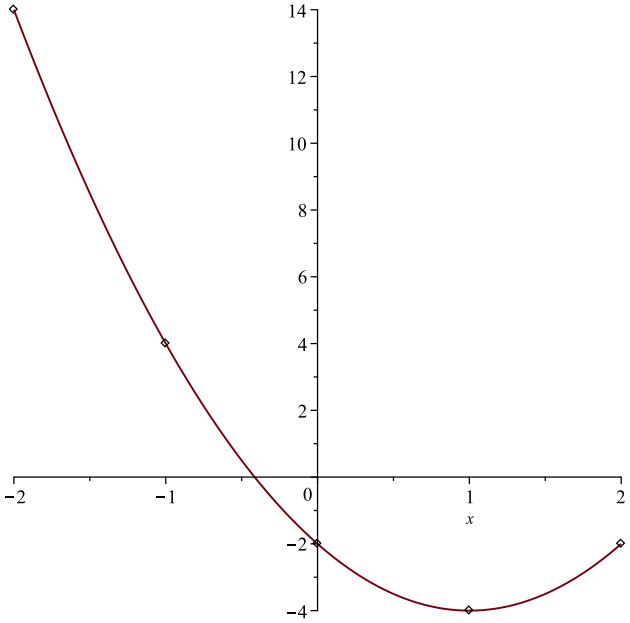
Notice as well that the left and right sides are mirrors of each other. Specifically they are mirrored over the line through the vertex known as the **line of symmetry**. In this case that is the vertical line $x = 0$.

The following example combines three modifications of the basic quadratic. We will graph it, notice the differences, and then use an activity to understand how each part of the equation produces a particular change in the graph.

Example 5.4.2 We will graph $y = 2(x - 1)^2 - 4$. First we complete this table

x	$2(x-1)^2-4$	
-2	$2(-2-1)^2-4$	$=14$
-1	$2(-1-1)^2-4$	$=4$
0	$2(0-1)^2-4$	$=-2$
1	$2(1-1)^2-4$	$=-4$
2	$2(2-1)^2-4$	$=-2$

Next we graph these points and sketch the graph through them.



Notice that this time the vertex is at $(1,-4)$. The line of symmetry is $x=1$. ▲

The following activity is a set of experiments to understand how each part of the parabola equation works. It is designed to help you understand by seeing the results in calculations. Do these exercises by hand. While you can look up the meaning of each equation part, memorizing them will be harder (and your ability to use them in later chapters reduced) if you do not see how each transformation is caused by the arithmetic.

Activity 5.4.3 Discovering Parabola Parameters.

(a) In this task we will check the effect of multiplying a quadratic by a scalar (number).

(i) Complete the table of points below. Compare the results to the table in Example 5.4.1

x	$2x^2$
-2	$2(-2)^2=8$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $2x^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(b) In this task we will continue to check the effect of multiplying a quadratic by a scalar (number).

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	$\frac{1}{2}x^2$
-2	$\frac{1}{2}(-2)^2 = 2$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $\frac{1}{2}x^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(c) In this task we will check on the effect of multiplying a quadratic by a negative scalar.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	$-x^2$
-2	$-(-2)^2 = -4$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $-x^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(d) In this task we will check on the effect of adding inside the square.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	$(x-1)^2$
-2	$(-2-1)^2 = 9$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $(x-1)^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(e) In this task we will continue to check on the effect of adding inside the square.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	$(x+1)^2$
-2	$(-2+1)^2 = 1$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $(x+1)^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(f) In this task we will check on the effect of adding outside the square.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	x^2-1
-2	$(-2)^2-1=3$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the x^2-1 column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(g) In this task we will continue to check on the effect of adding outside the square.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	x^2+1
-2	$(-2)^2+1=5$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the x^2+1 column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

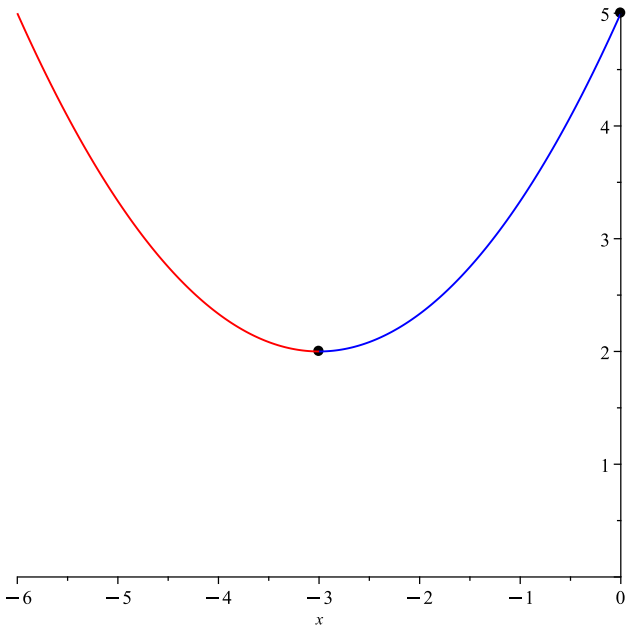
Example 5.4.4 Graph the parabola $y=\frac{1}{3}(x+3)^2+2$ by identifying the vertex and one other point.

From the activity we know that the vertex location is determined by the numbers added inside and outside the square. In this case the vertex is at (-3,2), because this is $(x-[-3])^2+[2]$.

The second point can be any point. $x=0$ is convenient. $y=\frac{1}{3}(0+3)^2+2=\frac{1}{3}\cdot 9+2=3+2=5$.

Because the center (axis of symmetry) is $x=-3$, we want to plot a portion of the curve left and right of that value. x from -6 to 0 (3 left to 3 right) will be convenient.

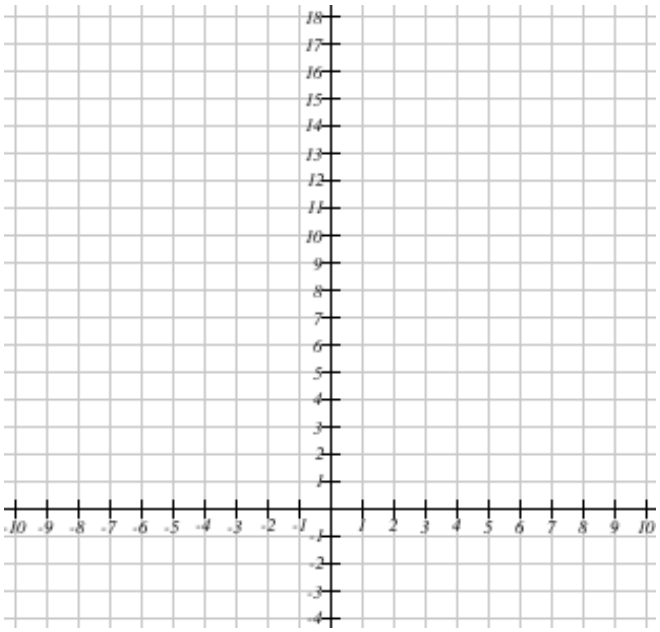
We can plot the two points we calculate and then sketch the portion of the curve to the right. The portion to the left we then sketch to be symmetrical.



▲

Checkpoint 5.4.5 Graph the parabola $y = 4(x+2)^2 + 2$

The vertex is _____



Recall that lines can all be expressed in the form $y = mx + b$ where m, the slope, defines steepness, and b represents a shift. Based on the work above we now know that all quadratics (parabolas) can be written as

$$y = a(x-h)^2 + k$$

where a indicates how steeply the parabola rises, and h and k indicate horizontal and vertical shifts of the parabola. The shifts are also the location of the vertex. In the example above we used the shifts to identify the coordinate of the vertex. In applications the shift parameters can have meaning.

5.4.2 Applications

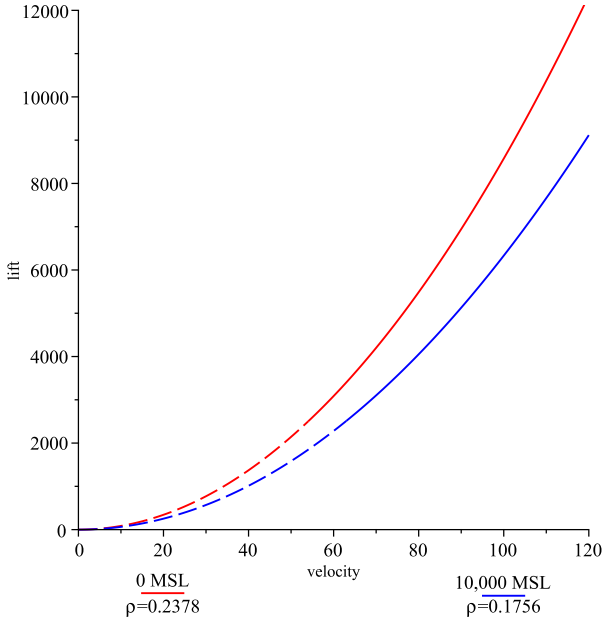
Now that we understand what each part of the equation of a parabola tells us, we can use that to interpret quadratic models and answer some questions using a quadratic model.

Example 5.4.6 The lift equation is a parabola. Note, only the right side of the parabola is part of the model, because negative speeds do not apply.

The lift equation has $h = 0$ and $k = 0$. This makes sense because there should be zero lift when there is zero speed. If $k > 0$ then there would be lift at zero speed. If $h > 0$ there would be no lift until that speed.

The coefficients, $\frac{1}{2}\rho S C_L$, make the parabola steeper. This means than increased air density, wing surface area, or coefficient of lift make the increase in lift occur faster with respect to increased speed.

For example, for a specific aircraft the increase in lift at ground level is steeper than at 10,000 MSL (air density given below).



▲

Example 5.4.7 The maximum load factor can be written as

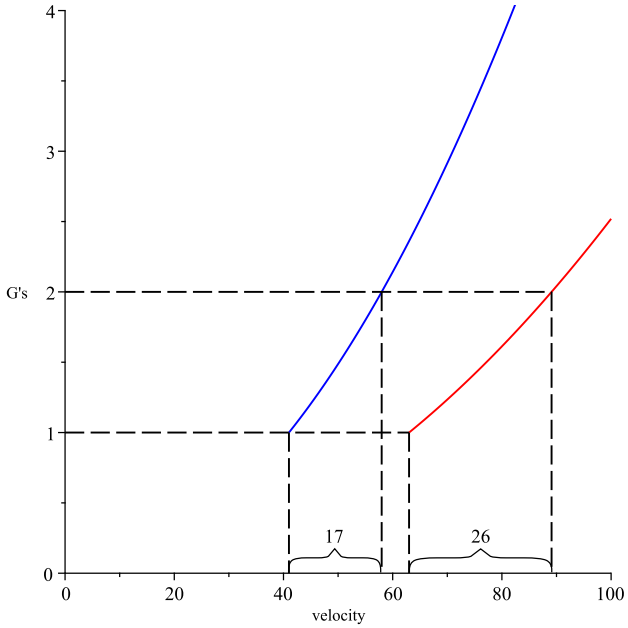
$$n_{max} = \left(\frac{V}{V_S}\right)^2 = \frac{1}{V_S^2}V^2.$$

This is a parabola.

This equation has $h = 0$ and $k = 0$. Note the stall speed of a fixed wing aircraft is the slowest speed at which it can fly. Thus the smallest number we will input into the model is $V = V_S$ which gives us 1 G. That is, the normal state of flight is 1 G (the one produced by earth's gravity). If $k > 0$ the model would indicate more than 1 G produced by gravity. $h \neq 0$ would indicate this occuring at some other speed.

The coefficient $(1/V_S^2)$ indicates that a higher stall speed (divide by a bigger number) will make the increase in G's slower (graph is less steep). This means it takes a greater increase of speed to move from 1 G to 2 G's than it does to move from 2 G's to 3 G's.

For example if the stall speed is 41 nm/hr, then 1 G occurs at 41 nm/hr and 2 G's occurs at $V = 41\sqrt{2} \approx 58$ nm/hr. To increase the maximum load factor requires an increase of $58 - 41 = 17$ nm/hr. However if the stall speed were 63 nm/hr then 1 G occurs at 63 nm/hr and 2 G's occurs at $V = 63\sqrt{2} \approx 89$ nm/hr. To increase the maximum load factor required an increase of $89 - 63 = 26$ nm/hr. The higher stall speed implies a greater $(22 > 16)$ increase in speed.



▲

The next example shows how we can determine the vertex and axis of symmetry even when the equation is not in the easy to read form. Here we use a method that does not require additional algebraic techniques. If you know how to complete the square, that will also work.

Example 5.4.8 Determine the coordinates of the vertex of $y = 2x^2 - 11x + 12$. All numbers are exact.

Because of the symmetry of a parabola we know that the vertex lies half way between any pair of points with the same height. For example it is half way between two points with $y = 0$, that is half way between the solutions to $2x^2 - 11x + 12 = 0$. We can solve this using the quadratic formula.

$$\begin{aligned} x &= \frac{-(-11) \pm \sqrt{(-11)^2 - 4(2)(12)}}{2(2)} \\ &= \frac{11 \pm \sqrt{121 - 96}}{4} \\ &= \frac{11 \pm \sqrt{25}}{4} \\ &= \frac{11 \pm 5}{4} \\ &= 4, 1.5 \end{aligned}$$

To calculate what is half way in between we average these two values.

$$\begin{aligned} x &= \frac{4 + 1.5}{2} \\ &= \frac{5.5}{2} \\ &= 2.75. \end{aligned}$$

Thus the x-coordinate of the vertex is 2.75. To calculate the y-coordinate we substitute the x value into the quadratic.

$$2(2.75)^2 - 11(2.75) + 12 = -3.125.$$

▲

Simplified Effect of Gravity. A model of motion due to gravity is beyond the scope of this course, but we can understand a connection to quadratics. Gravity is a force that accelerates objects toward each other. The strength of earth's gravity at the surface is approximately 9.80665 $\frac{\text{m}}{\text{s}^2}$. Be aware that this value actually varies a little by location and decreases with altitude, but those effects are not needed for any problems in this book.

Notice that gravity is an acceleration (i.e., m/s is a velocity and this is (m/s)/s meaning a change in velocity) and as a result the time portion is squared. This means accleration due to gravity is quadratic with respect to time.

For objects that are dropped, we can figure out both their speed and their height can be determined.

Example 5.4.9 Gravity's Effect on Velocity. If we drop a ball from 1.0 meters above the surface, it will start with a velocity of 0 m/s. Acceleration due to gravity is (9.8 m/s)/s which means after one second the ball will be travelling down at 9.8 m/s. After another second (2 seconds total) it would increase (accelerate) another 9.8 m/s so its velocity would be 18.6 m/s.

We can write an equation for the velocity after t seconds. Notice that the change in velocity is a constant, so the change in velocity is a linear equation. The rate (slope) is 9.80665 $\frac{\text{m}}{\text{s}^2}$. The shift is the initial velocity (0 m/s in this case).

$$v = 0\ \frac{\text{m}}{\text{s}} + \frac{9/8\ \text{m/s}}{\text{s}}(t\ \text{s})$$

▲

Calculating the exact height of an object falling due to gravity requires mathematics beyond this textbook. A little more information can be found in Section 5.5. The following example illustrates calculations using a provided simplification.

Example 5.4.10 Falling Due to Gravity. Suppose the height of an object falling due to gravity is given by

$$h = -4.9t^2 + 1.0.$$

(a) Determine when the object strikes the ground. The ground is at height 0.

We can determine this by solving the following equation.

$$\begin{aligned} -4.9t^2 + 1.0 &= 0. \\ -4.9t^2 &= -1.0. \\ t^2 &= \frac{-1.0}{-4.9}. \\ t^2 &\approx 0.2\underline{0}40816327. \\ t &\approx \sqrt{0.2\underline{0}40816327}. \\ t &\approx 0.4\underline{5}17539515 \\ t &\approx 0.45 \end{aligned}$$

It will take 0.45 seconds for the object to reach the ground.

(b) Determine the height from which it was dropped.

Because the object is dropped, the heighest point is when it is dropped which is time $t = 0$. Thus the height from which it was dropped is $-4.9(0) + 1.0 = 1.0$ meters.

▲

Example 5.4.11 Gravity's Effect on a Thrown Object. Consider an object that is thrown upward. Gravity will first slow down the object as it climbs, then it will cause the object to fall (as in the previous example). Suppose the height of this object is given by

$$h = -4.9t^2 + 5.1t + 1.2.$$

(a) Determine when the object strikes the ground. The ground is at height 0.

We can determine this by solving the $-4.9t^2 + 5.1t + 1.2 = 0$. To do this we can use the quadratic formula. All numbers have two significant digits and are precise to the 10ths.

$$\begin{aligned}
t &= \frac{-(5.1) \pm \sqrt{(5.1)^2 - 4(-4.9)(1.2)}}{2(-4.9)} \\
&= \frac{-5.1 \pm \sqrt{26.01 + 23.52}}{-9.8} \text{ Products maintains 2 sigfigs} \\
&= \frac{-5.1 \pm \sqrt{49.53}}{-9.8} \text{ Sum maintains to units} \\
&\approx \frac{-5.1 \pm 7.037755324}{-9.8} \text{ Root maintains 2 sigfigs} \\
&\approx \frac{1.9377553 24}{-9.8}, \frac{-12.137755324}{-9.8} \text{ Sum maintains to 10ths} \\
&\approx -0.1977301351, 1.238546461 \text{ Division maintains 2 sigfigs} \\
&\approx -0.20, 1.2
\end{aligned}$$

The negative time does not make sense, so $t \approx 1.2$ seconds is when the object will reach the ground.

(b) Determine the maximum height the object reached.

This will occur at the vertex of the parabola. Because the equation is not in the easy form we will use the technique demonstrated in Example 5.4.8.

We use any two points. In the previous step we found the two times when the height is zero. It does not matter that one is negative: this will still work to locate the vertex. The time at which the maximum height occurs is

$$\frac{-0.1977301351 + 1.238546461}{2} \approx 0.520408163 \approx 0.52.$$

The height at that time is

$$\begin{aligned}
-4.9(0.520408163)^2 + 5.1(0.520408163) + 1.2 \approx & \text{ Products maintain 2 sigfigs} \\
-1.327040815 + 2.654081631 + 1.2 \approx & \text{ Sum maintains to 10ths} \\
2.527040816 \approx 2.5 &
\end{aligned}$$

The object will reach a maximum height of 2.5 meters at 0.52 seconds after it is thrown.

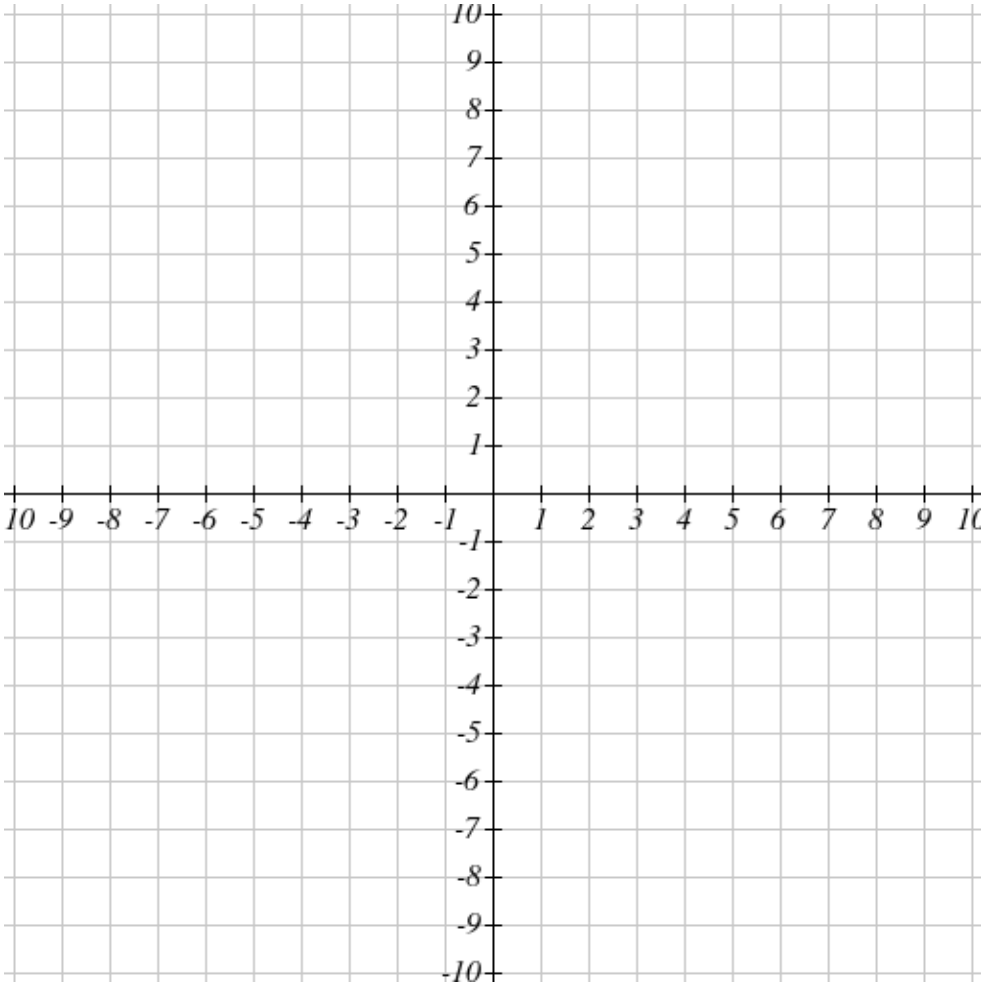
▲

5.4.3 Exercises

Graphing Quadratics. Graph each of these equations. Use your knowledge of translations and stretches.

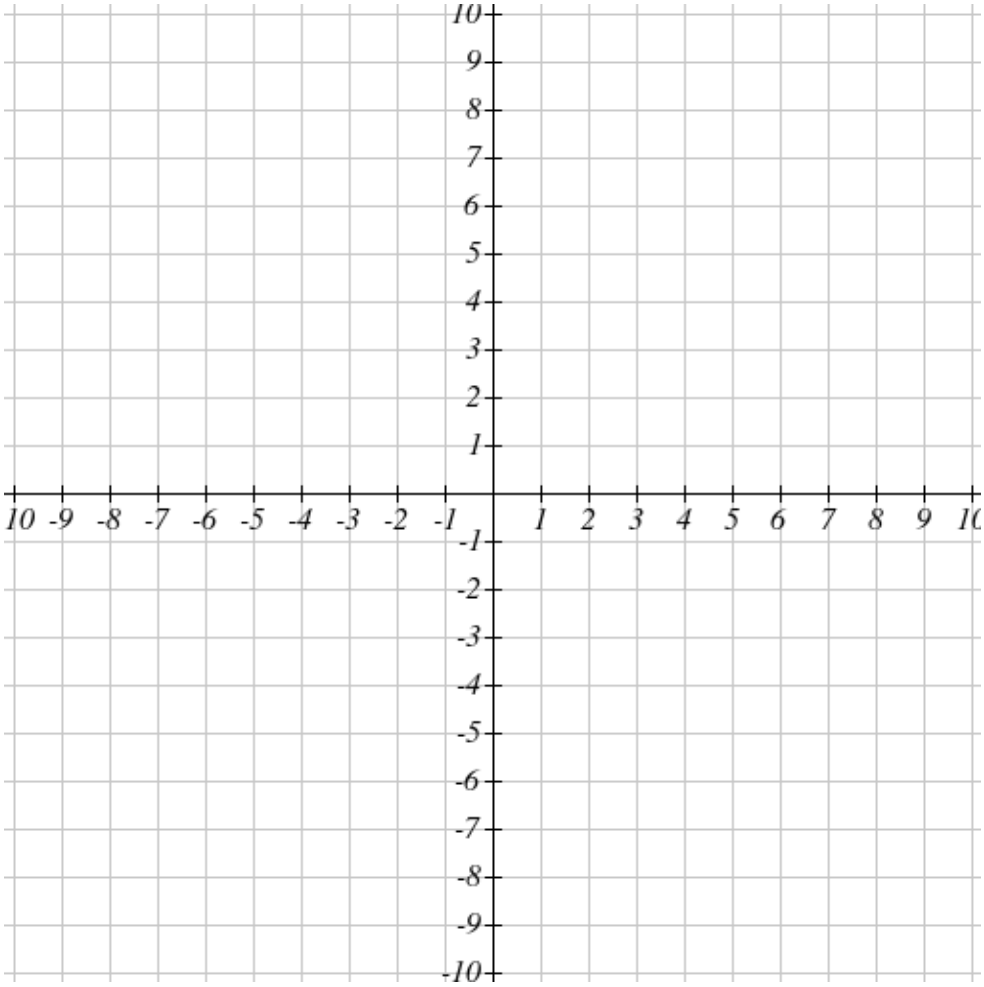
1. Graph the parabola: $f(x) = (x-3)^2$

The vertex is _____



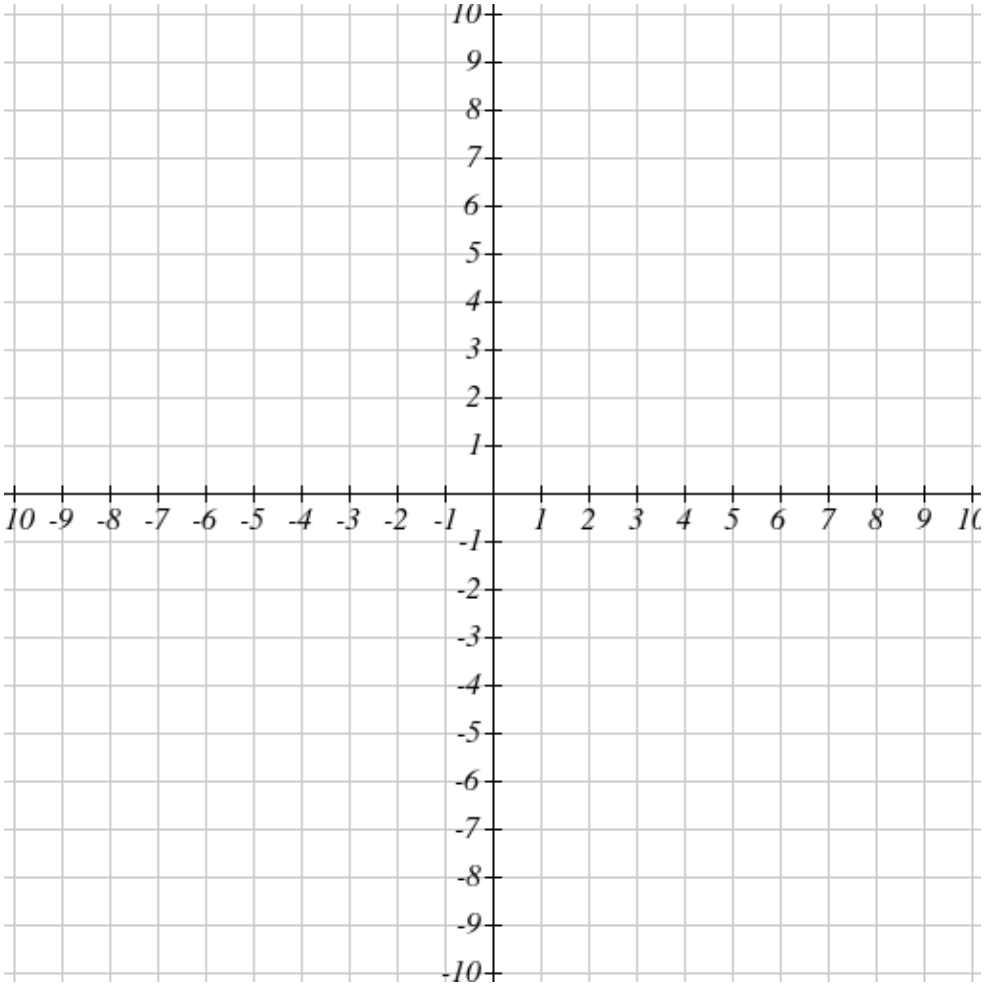
2. Graph the parabola: $f(x) = (x-3)^2 + 1$

The vertex is _____



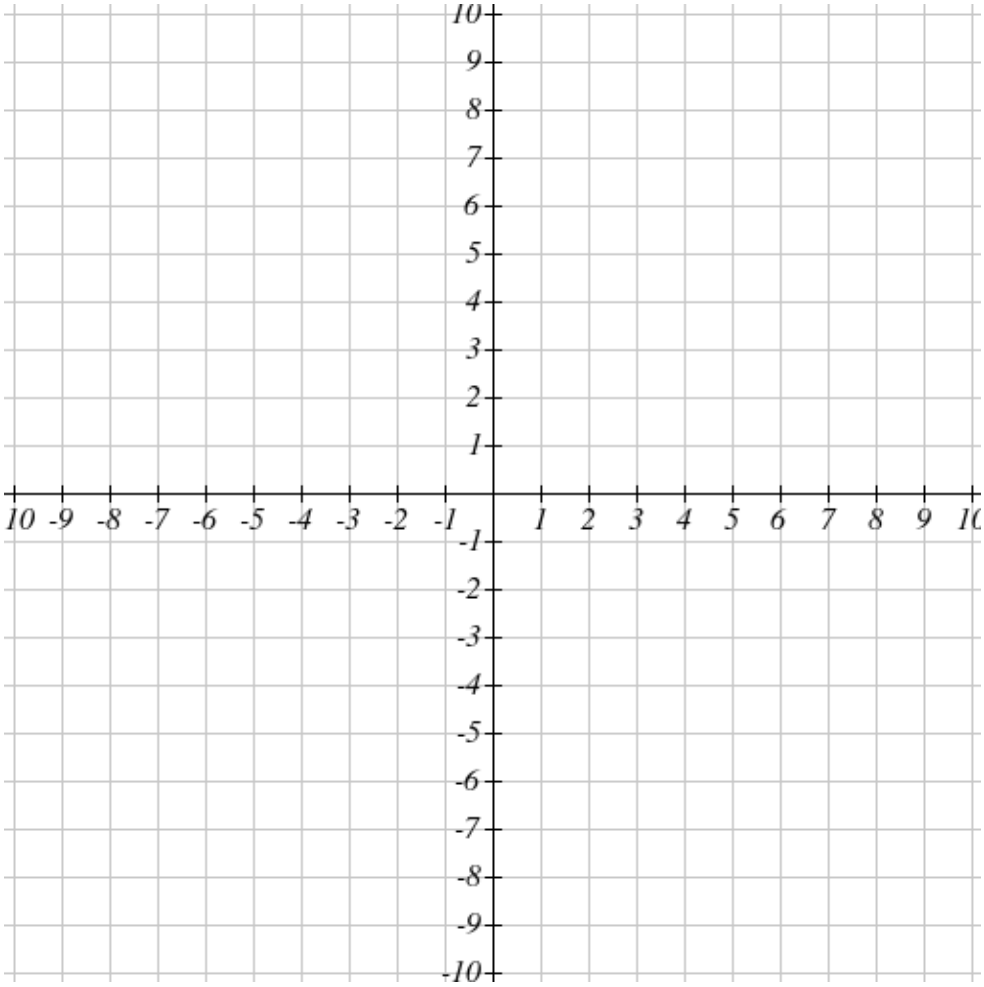
3. Graph the parabola: $f(x) = 4x^2$

The vertex is _____



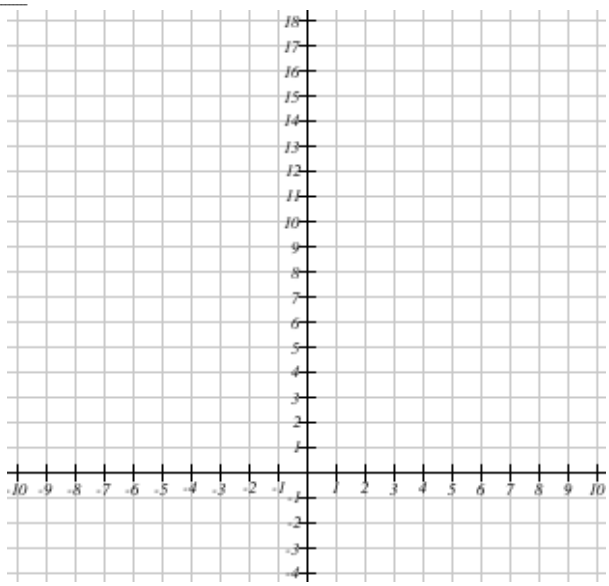
4. Graph the parabola: $f(x) = \frac{1}{4}x^2$

The vertex is _____



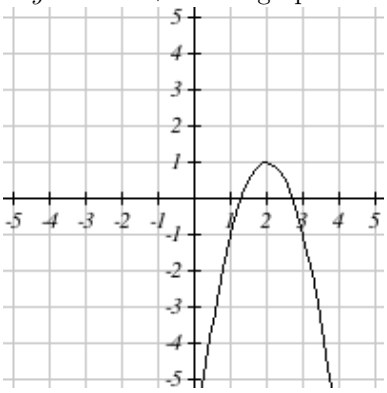
5. Graph the parabola $y = 4(x+2)^2 + 2$

The vertex is _____



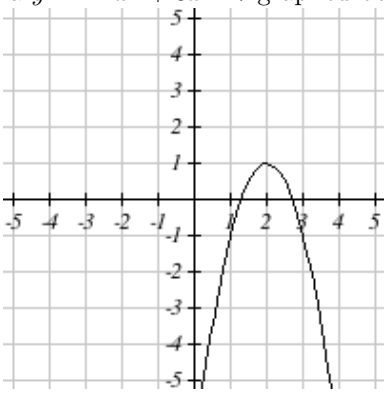
Reading Quadratic Graphs. Use the provided graph to answer these questions about quadratics.

6. Identify the vertex of the parabola $y = -2x^2 + 8x - 7$ graphed below



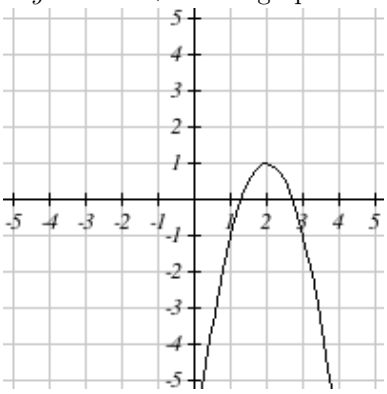
Vertex = (__, __)

7. Identify the vertex of the parabola $y = -2x^2 + 8x - 7$ graphed below



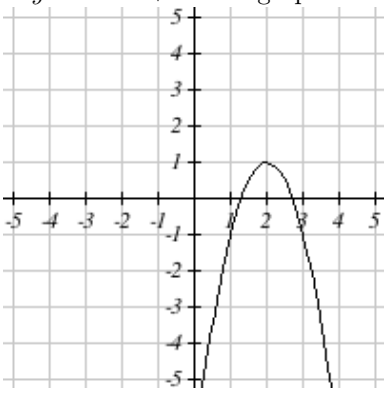
Vertex = (__, __)

8. Identify the vertex of the parabola $y = -2x^2 + 8x - 7$ graphed below



Vertex = (__, __)

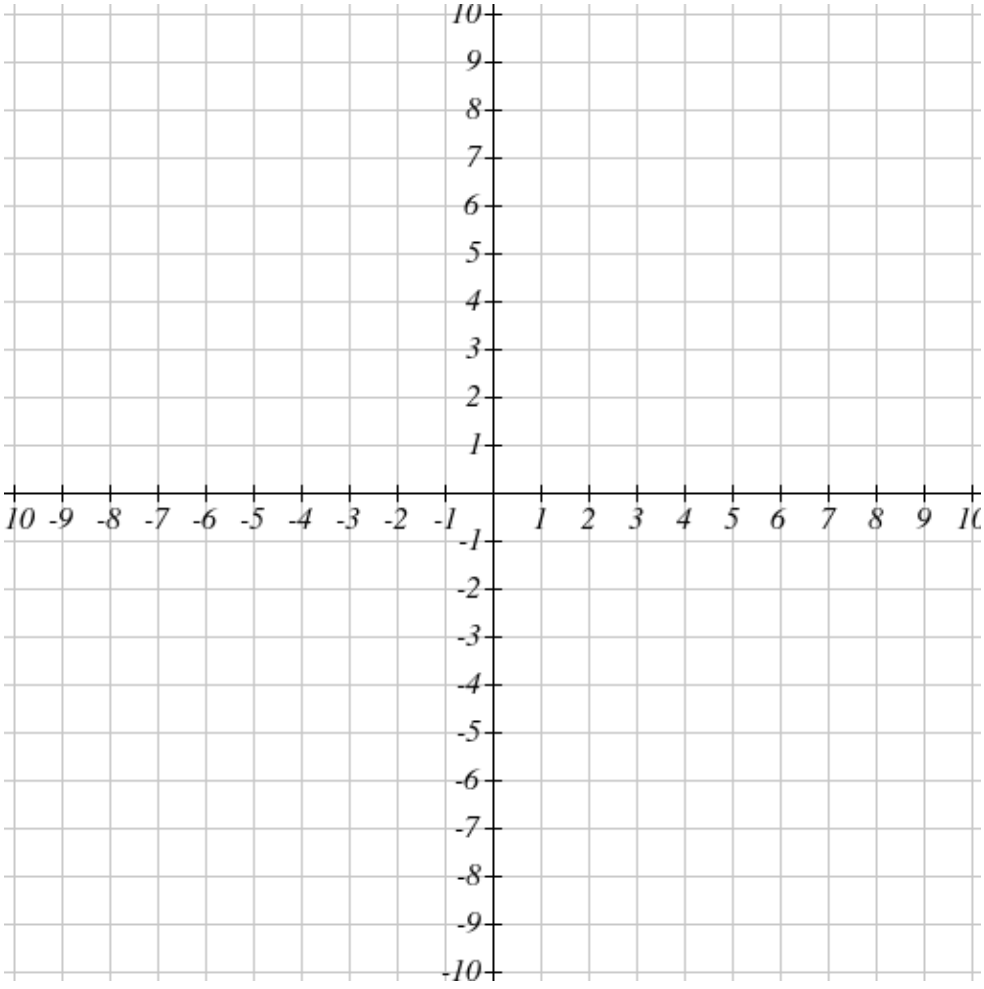
9. Identify the vertex of the parabola $y = -2x^2 + 8x - 7$ graphed below



Vertex = (__, __)

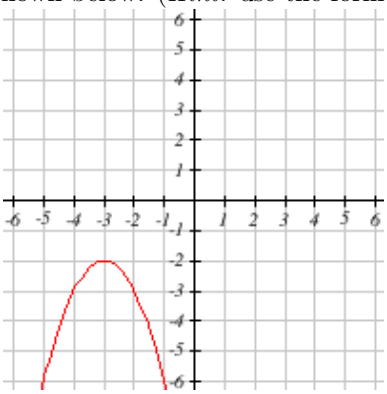
10. Graph the parabola: $f(x) = -2x^2 - 12x - 15$

The vertex is _____



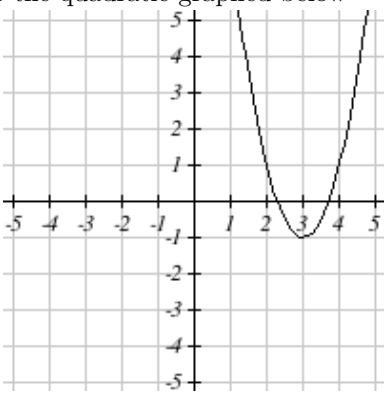
Write Equations for Quadratics. Write a quadratic equation to match the graph or information provided.

11. Find an equation for the graph shown below. (*Hint*: use the form $y = a(x-h)^2 + k$)



$y =$ _____

12. Write an equation (any form) for the quadratic graphed below



y = _____

13. A quadratic function has its vertex at the point $(-9, 4)$. The function passes through the point $(2, 1)$. Find the expanded form of the function.

The coefficient a is _____ .

The coefficient b is _____ .

The constant term c is _____ .

Exercise Group. Use the nature of quadratic graphs (parabolas) to answer these questions.

14. NASA launches a rocket at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h(t) = -4.9t^2 + 115t + 94$.

(A) Assuming that the rocket will splash down into the ocean, at what time does splashdown occur? (Round answer to 2 decimal places)

The rocket splashes down after _____ seconds.

(B) How high above sea-level does the rocket get at its peak? (Round answer to 2 decimal places)

The rocket peaks at _____ meters above sea-level.

15. Methane gas is produced in a variety of ways including by landfills, natural gas systems, and coal mines. Based on data from the U.S. EPA, projected methane emissions in the U.S. can be modeled by the function

$m(t) = -0.072t^2 + 1.93t + 173.9$

where

$m(t)$ is the amount of methane in units of million metric tons

t is the number of years after 2000.

What is the predicted amount of methane in 2008? _____

In what year does this predict maximum methane emissions? _____
How much methane will be produced that year? _____
Round all answers to units. The model inputs (years after 2000) must be integers.

5.5 Project: Approximation of Sums

Project 5.5.1 Approximate Height of a Dropped Object. Using the acceleration due to gravity, $9.80665 \frac{\text{m}}{\text{s}^2}$, we can calculate the velocity at any time. However, we can only approximate its position, because the velocity is constantly changing. This activity demonstrates a method of approximation that is used in many problems including calculating areas (e.g., calculating the amount of asphalt needed for a driveway with curved sides). This activity also illustrates how approximations depend on choices we make.

(a) Review the process shown here.

Suppose a ball is dropped from a height of 60.0 meters. We will approximate its height above ground by calculating its approximate position each second. For example, after 1 second the ball will have accelerated from 0 m/s to 9.8 m/s. If we pretend that it was travelling 9.8 m/s the whole time, then we estimate it will be 9.8 m lower than it began. This is a height of $60.0 - 9.8 = 50.2$. A second later the velocity will have increased 9.8 m/s so it will be $9.8 + 9.8 = 19.6$ m/s. If we pretend that it was travelling 19.6 m/s the whole time, then we estimate it will be 19.6 m lower than it began. This is a height of $50.2 - 19.6 = 30.6$. Another second later (third second) the velocity will have increased by 9.8 m/s so it will be $19.6 + 9.8 = 29.4$. If we pretend that it was travelling 29.4 m/s the whole time, then we estimate it will be 29.4 m lower than it began. This is a height of $30.6 - 29.4 = 1.2$. With only 1.2 meters left and a velocity much greater than 1.2, we know it will reach the ground in less than one more second.

The table below contains these results.

Time	Velocity	Distance Traveled	Height
0	0	0	60.0
1	9.8	9.8	50.2
2	19.6	19.6	30.6
3	29.4	29.4	1.2

(b) Repeat this exercise using half second intervals instead of one second intervals. Remember that in a half second the acceleration will be $9.8/2 = 4.9$ m/s. In the first half second the ball will fall $0.5 \text{ s} \cdot 4.9 \frac{\text{m}}{\text{s}} = 2.45$ m. In the second half second the velocity would be 9.8 m/s so it would drop $0.5 \text{ s} \cdot 9.8 \frac{\text{m}}{\text{s}} = 2.45$ m. The first two entries are completed for you. Add more rows if you need to.

Time	Velocity	Distance Traveled	Height
0	0	0	60.0
0.5	4.9	2.45	57.55
1.0	9.8	4.9	52.65
1.5			
2.0			
2.5			
3.0			

(c) The end result of this task will be a third table calculated in 1/4 second intervals.

(i) What is the acceleration in 1/4 second? Recall it is 9.8 m/s which was 4.9 m/half second.

(ii) Fill out a table in 1/4 second intervals.

(d) Notice that in the first table (1 second intervals), we estimated after one second the height would be 50.2 m. In the second table (1/2 second intervals), we estimated after one second the height would be 52.65 m.

(i) Find other times in which the 3 tables have different estimates.

(ii) Are the estimates in one table always less than or greater than those in the other tables?

(iii) Explain why these estimates are different.

(iv) Which estimate do you think is most accurate?

Chapter 6

Exponential

6.1 Exponential Relations

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Read and interpret models (critical thinking)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

Section 3.3 presented different rates at which data changes including linear, quadratic, and exponential. This section presents models that require exponentials and notation so we can perform calculations. The ability to solve equations with exponentials comes in Section 6.4.

6.1.1 Comparing Growth Rates

In colloquial speech **exponential** is used to mean "very fast". In this text we are using the term formally which requires a more detailed definition. This section presents examples that illustrate that an exponential relation is very fast in that it grows faster than linear and quadratic relations. To complete the presentation of comparative rates we will also show another relation that is even faster. Thus we can be more precise about how fast something grows.

Consider the linear, quadratic, and exponential relations in Table 6.1.1, Table 6.1.2, and Table 6.1.3 respectively. Linear data has the same differences (3 in the example below). In contrast the differences for quadratic and exponential data grow as the data grows. Their growth is not the same however. Quadratic differences grow at the same rate (2nd differences are linear, 4 in the example below). In contrast exponential differences grow faster as the data grows faster (indeed at the same rate).

Another way to look at this, is that for linears and quadratics eventually the data is bigger than the differences. For example 5,8,11 are all bigger than the difference 3 for the linear below. Likewise, the values 8, 18, and 32 are greater than the differences 6, 10, 14 respectively for the quadratic below. For exponential relations, however, the rate of growth of the values is a multiple of the values. The rate of growth increases as quickly as the values do. For example the values 6, 12, 24 are exactly equal to the differences 6, 12, 24. In this example the values are doubling and so are the differences between them. Because of this exponential growth will always outpace linear and quadratic growth eventually.

Table 6.1.1 Linear Relation

n	0	1	2	3	4	5
$3n+2$	2	5	8	11	14	17
Differences		3	3	3	3	3

Table 6.1.2 Quadratic Relation

n	0	1	2	3	4	5
$2n^2$	0	2	8	18	32	50
Differences		2	6	10	14	18

Table 6.1.3 Exponential Relation

n	0	1	2	3	4	5
$3(2^n)$	3	6	12	24	48	96
Differences		3	6	12	24	48

We can also compare the ratios of consecutive terms of the relations. Because the ratio can be converted to a percent, we can also think of the growth as percent increase. Table 6.1.4 and Table 6.1.5 have the ratios for a linear and a quadratic relation respectively. From the decimal approximation rows we can see that the ratios are decreasing. A closer look shows they are both trending toward one. In contrast in Table 6.1.6 we are reminded that the ratios for an exponential relation are constant. This also implies that exponential relations grow faster than linear or quadrataic relations.

Table 6.1.4 Linear Relation

n	0	1	2	3	4	5
$3n+2$	2	5	8	11	14	17
Ratios		5/2	8/5	11/8	14/11	17/14
Percent		250%	160%	137.5%	127.27%	121.43%
Percent Increase		150%	60%	37.5%	27.27%	21.43%

Table 6.1.5 Quadratic Relation

n	1	2	3	4	5	6
$2n^2$	2	8	18	32	50	72
Ratios		8/2	18/8	32/18	50/32	72/50
Percent		400%	225%	177.78%	156.25%	144%
Percent Increase		300%	125%	77.78%	56.25%	44%

Table 6.1.6 Exponential Relation

n	0	1	2	3	4	5
$3(2^n)$	3	6	12	24	48	96
Ratios		2	2	2	2	2
Percent		200%	200%	200%	200%	200%
Percent Increase		100%	100%	100%	100%	100%

The comparisons to linear and quadratic might suggest that exponential is the fastest growing relation. It is not. As one example of a faster relation consider Table 6.1.7 which is known as factorial. Interestingly, the ratios grow linearly. This is vaguely like quadratic relations which grow faster than a linear relation because their differences grow linearly. Factorial relations are growing faster than exponential relations because their ratios are growing linearly rather than remaining constant.

Table 6.1.7 Factorial Relation

n	0	1	2	3	4	5
$n!$	1	1	2	6	24	120
Ratios		1	2	3	4	5

Checkpoint 6.1.8 Rank the following based on how fast they grow.

a 0 1 32 243 1024 3125

b 256 320 400 500 625 781.25

c -1 7 15 23 31 39

1. a
2. b
3. c

1. a
2. b
3. c

1. a
2. b
3. c

6.1.2 Applications

This section presents some simple applications that introduce us to the nature of exponential relations and introduce us to how to write equations for exponential models. The next few examples are used to present the mathematics, they are not a complete presentation of the science.

Bacteria (any cells) grow by each cell dividing into two, completely functioning cells. The new cells eventually reproduce by dividing in half as well. This means that the population is doubling. Because each cell takes roughly the same amount of time to grow enough to be able to divide, the population will double again when that much time has elapsed. Because we are interested in the idea, we will not round in the next few examples.

Example 6.1.9 The bacteria **lactobacillus acidophilus** is part of turning milk into yogurt. Based on experiments a new generation of bacteria are formed every 70 minutes, that is, the population doubles every 70 minutes.

Suppose when we start tracking the data there are 4000 cells. After 70 minutes, all of these will have divided into two, so there will be $4000(2) = 8000$ cells. After another 70 minutes, all of these 8000 cells will have divided into two, so there will be $8000(2) = 16000$ cells. These results are shown in Table 6.1.10.

If the population grows by the same ratio every 70 minutes, we might ask ourselves if it grows by the same ratio every 35 minutes. Using math from later in this chapter we can construct Table 6.1.11 which shows populations every 35 minutes. We discover that the population does grow by a fixed percent every 35 minutes. If we look a little deeper we realize that we cut the time in half and switched from doubling (times 2) to times $\sqrt{2} \approx 1.4142$.

We could produce a table for any amount of time (e.g., every hour) and we would discover that the population grows by the same multiple every time. ▲

Table 6.1.10 Growth of Lactobacillus Acidophilus

Minutes	0	70	140	210
Population	4000	8000	16000	32000
Increased by		4000	8000	16000

Table 6.1.11 Growth of Lactobacillus Acidophilus

Minutes	0	35	70	105	140
Population	4000	5657	8000	11314	16000
Ratio		1.4142	1.4142	1.4142	1.4142

Having emphasized that exponential data grows by a fixed ratio (multiple) every time unit, we can now work toward mathematical notation.

Example 6.1.12 First Exponential Model. To obtain a model (equation) we will review our calculations from the previous example and note a pattern in those calculations.

The initial amount (information we are given) is $P_0 = 4000$. After 70 minutes all of these split into two so the population is $P_{70} = 4000(2) = 8000$. After 140 minutes they have all split into two a second time so the population is $P_{140} = 4000(2)(2) = 16000$. After 210 minutes they have split a third time increasing the population to $P_{210} = 4000(2)(2)(2) = 32000$. We repeatedly mutiply the initial amount by 2. This means we will have a power of 2 times the initial amount.

The number of 2's by which we multiply is determined by how many multiples of 70 minutes have expired. $140/70 = 2$ so we multiplied by 2^2. $210/70 = 3$ so we multiplied by 2^3. In general we multiply by two $t/70$ times.

Putting these together implies we want to multiply the initial amount (4000 in this case) by two (double the population) for each of multiple of 70. This gives us

$$P = 4000 \cdot 2^{t/70}.$$

▲

Example 6.1.13 First Exponential Model Redux. Of course we cannot actually count the number of cells in a colony of bacteria. It is easier to measure by mass (units of grams). If the number of cells has doubled then the mass will have about doubled as well.

Suppose 3 grams of lactobacillus acidophilus is placed in milk. What is a model for the mass of bacteria if it doubles every 70 minutes?

From the previous example we know we can multiply the initial amount (3 g) by 2 for each 70 minutes. This gives us

$$A = 3(2^{t/70}).$$

▲

Some exponential relations show the amount decreasing. That is the ratio that is multiplied is between 0 and 1. One example is radioactive decay. Radioactive substances are not stable. The radiation they give off is the result of the atoms breaking down into other substances. This means that over time the amount of the radioactive substance decreases. It has been shown that this decrease is exponential. The rate of decay is expressed in **half-life**, the amount of time for the substance to be reduced by half.

Example 6.1.14 Barium-133 has a half-life of 10.551 years. This means that afeter 10.551 years only half of the original amount will remain. This is the result of the radioactive isotope breaking down into other substances.

Suppose that we obtain 7.000 grams of Barium-133. After 10.551 years we will have $7.000 \cdot \frac{1}{2} = 3.500$ grams. Note the 1/2 is an exact number (the measurement error is in the number of years). After 21.102 years there will be $7.000 \cdot \left(\frac{1}{2}\right)^2 = 1.750$. As before the amount left after t years can be written as the initial amount times the ratio for each 10.551 years.

$$A = 7.000 \left(\frac{1}{2}\right)^{t/10.551}.$$

Using a algebraic notational change ($2^{-1} = 1/2$), this could also be written as

$$\begin{aligned} A &= 7.000 \cdot \left(\frac{1}{2}\right)^{t/10.551} \\ &= 7.000 \cdot \left(2^{-1}\right)^{t/10.551} \\ &= 7.000 \cdot 2^{-t/10.551}. \end{aligned}$$

The negative exponent expresses that the process is decreasing, that is, the amount is going down. ▲

Exponential relations are not all doubling or cutting in half. The ratio can be anything. Going viral on the internet could be (but is not always) an exponential growth of views.

Example 6.1.15 A new cat video is posted and 12 people view it the first day. Every 4 days afterward the number of people who see it triples. Write an equation to model the total number of people who have viewed this video.

To help ourselves figure this out we can calculate the first few days' results. The 4 days after the video is posted, there will be $12(3) = 36$ views. The eighth day, there will be $12(3)(3) = 108$ views. This suggests that for each additional four days we multiply the result by 3.

Thus we need to divide the number of days by 4 to determine how many times it has tripled. For example after 24 days we expect it to triple $24/4 = 6$ times.

To calculate the total number of views we multiply the original twelve views by 3 for each time it tripled. Thus the number of people viewing the video is

$$v = 12(3^{d/4})$$

▲

Checkpoint 6.1.16 At a particular temperature and acidity bacillus megaterium will doube every 0.62 hours. Suppose we begin with 1.0 grams of the bacteris. Write an equation for the amount of bacteria after t hours.

$b =$ _____

Use t as the variable.

We learned in Subsection 3.3.2 that salaries increased by a fixed percent each year are exponential in nature. Now we can write a model for this and calculate results.

Example 6.1.17 Tien's initial salary was \$52,429.33. He received a 5% raise each year. What should Tien's salary be entering the sixth year?

Because the raise is a 5% increase, each year's salary is 105% of the previous years. In decimal the percent is $p = 1.05$. Thus after one year his salary will be $\$52,429.33(1.05)$. After two years his salary will be $\$52,429.33(1.05)(1.05)$. This pattern will continue.

The model then initial salary time 1.05 for each year.

$$S = \$52,429.33(1.05)^t$$

where t is the number of years since he was hired. Entering the sixth year would mean he has just received his fifth raise. His salary would be

$$S = \$52,429.33(1.05)^5 = \$66,914.59.$$

▲

Example 6.1.18 If Moses' salary after six raises was \$72,311.54, and he received a 4% raise each year. What was his initial salary?

Because the raise is a 4% increase, the percent is $p = 1.04$. The model then is each year's salary is the original salary times 1.04 for each raise.

$$S = S_0(1.04)^t$$

where t is the number of years since he was hired, and S_0 is the initial salary. We can now solve for the

initial salary.

$$\begin{aligned}\$72,311.54 &= S_0(1.04)^6.\\ \$72,311.54 &\approx S_0(1.2653).\\ \frac{\$72,311.54}{1.2653} &\approx \frac{S_0(1.2653)}{1.2653}.\\ \$57,149.72 &\approx S_0.\end{aligned}$$

Because the salary would be rounded each year this might be off by a small amount, but not enough to matter for our curiosity. ▲

In the previous example we rounded in the second step to 4 decimal places. Was this the right choice? We can test by trying the calculation with fewer decimal places. If we had rounded to 5 places, the last step would have been $\frac{\$72,311.54}{1.26532} \approx \$57,148.82$. If we had rounded to 6 places, the last step would have been $\frac{\$72,311.54}{1.265319} \approx \$57,148.86$. If we had rounded to 7 places, the last step would have been $\frac{\$72,311.54}{1.265320} \approx \$57,148.86$. Likewise testing 8 places would not change the amount. Thus any more than 6 places is not enough to change the result. However, was the 4 places result bad? The amounts between the roundings are different from each other but not by enough to impact anyone's life. Remember rounding is often about being practical: if the variation we round away has no impact, why bother with all the extra work?

Because we just calculated an initial salary from a current salary and the annual percent increase, we might wonder if we can calculate the percent increase from a past and current salary. We can, but that calculation is not an exponential calculation. Rather it requires a technique for roots from Section 5.3.

Example 6.1.19 We will calculate the percent increase given initial and final salaries. If Raven's initial salary was \$53,242.17, and her salary at the end of 7 years was \$67,368.33, what was her annual percentage increase?

Solution. The end of seven years means there have been six raises. From the two data points we know

$$\begin{aligned}67368.33 &= 53242.17(r^6)\\ \frac{67368.33}{53242.17} &= r^6\\ \sqrt[6]{\frac{67368.33}{53242.17}} &= \sqrt[6]{r^6}\\ 1.04 &= r\end{aligned}$$

Thus her annual percentage increase was 4%. The full model is

$$S = 53242.17(1.04)^t$$

where t is time in years. ▲

6.1.3 Two Exponential Models

Above we worked examples in which some initial amount (e.g., mass of bacteria or salary) was repeatedly multiplied by a number. These were written in slightly different ways giving us the following two forms of the exponential model.

Model 6.1.20 Exponential Growth (rate). *An amount that is growing exponentially with respect to time is given by*

$$A = A_0\left(r^{t/d}\right)$$

where

- *A is the final amount (units vary)*
- *A_0 is the initial amount (same units as A)*

- *r is the doubling ($r = 2$), trebling ($r = 3$), or other increase. It is unitless.*
- *t is the amount of time (units typically days/hours/minutes)*
- *d is how long until the increase occurs (same units as t)*

Model 6.1.21 Exponential Growth (percent). *An amount that is growing exponentially with respect to time is given by*

$$A = A_0 (1 + p)^t$$

where

- *A is the final amount (units vary)*
- *A_0 is the initial amount (same units as A)*
- *p is the percent increase or decrease. It is unitless.*
- *t is the amount of time (units typically days/hours/minutes)*

It is possible to convert from one form to the other. The following examples demonstrate determining the percent increase/decrease for both forms which may require converting to the second forms. Percent increase and decrease may be reviewed in Subsection 2.1.2.

Example 6.1.22 Suppose $P = 100(1.032)^t$ where t is the number of years. What is the annual percent increase?

We know that

$$\begin{aligned} 1.032 &= 1 + p \\ 1.032 - 1 &= p \\ 0.032 &= p \end{aligned}$$

so this is a 3.2% increase. ▲

Example 6.1.23 Suppose $P = 100(0.88)^t$ where t is the number of years. What is the annual percent decrease?

We know that

$$\begin{aligned} 0.88 &= 1 + p \\ 0.88 - 1 &= p \\ -0.12 &= p \end{aligned}$$

Because p is negative this is a decrease. The annual percent decrease is 12%. ▲

Generally if the value $1+p$ is greater than one the exponential is increasing (represents a percent increase), and if the value is less than one, the exponential is decreasing (represents a percent decrease).

Example 6.1.24 Suppose $P = 230(1.03)^t$. Is this increase or decrease and what is the percent increase/decrease?

Solution. Because $1.03 > 1.00$ this is a percent increase.

$$\begin{aligned} 1.03 &= 1 + p \\ 1.03 - 1 &= p \\ 0.03 &= p \end{aligned}$$

Thus this is a 3% increase, because the percent increase/decrease is how much more/less than 100%. ▲

Even when the exponential model is expressed in terms of doubling or similar, we can determine the percent increase or decrease. This requires knowing the following property $2^{t/3} = (2^{1/3})^t \approx (1.2599)^t$ or in

general

$$b^{t/r} = (b^{1/r})^t.$$

Example 6.1.25 In the exponential model $P = 342\,(2)^{t/2.06}$ what is the percent increase or decrease? Note these numbers are measurements.

First, we use the property stated above to convert the form

$$\begin{aligned}(2)^{t/2.06} &= \\ \left(2^{\frac{1}{2.06}}\right)^t &\approx 1.400009768^t \\ &\approx 1.40^t\end{aligned}$$

Because 1.4 is bigger than one it is a percent increase. Note $1.40 = 1 + 0.40$, so this is a 40% increase, because the percent increase/decrease is how much more/less than 100%.

For rounding we used that 2 and 1 (in the division) are exact numbers. Calculating the power of 2 maintains the number of significant digits (3). When we subtract precision is maintained to the hundredths position (2 significant digits). ▲

Significant Digits beyond Arithmetic. In Significant Digits beyond Arithmetic we noted that a square root maintains the same number of significant digits. The same mathematics applies to exponential. For example, $2^{3.51} \approx 11.39240156 \approx 11.4$.

Example 6.1.26 In the exponential model $P = 342\,(2)^{-t/27}$ what is the percent increase or decrease?

First, we notice that the exponent is negative which we know means the amount is decreasing. To determine the percent decrease we use the algebra illustrated above to convert the form

$$\begin{aligned}(2)^{-t/27} &= \\ \left(2^{\frac{-1}{27}}\right)^t &= 0.9746546091^t \\ &\approx (0.97)^t\end{aligned}$$

Because 0.97 is less than one it is a percent decrease. Note $1 - 0.97 = 0.03$, so this is a 3% decrease, because the percent increase/decrease is how much more/less than 100%.

We round to two decimal places because 27 has 2 significant digits. The subtraction retains precision to the hundredths position which is only one significant digit. ▲

Checkpoint 6.1.27 In the model $A = 1900 \cdot \left(2^{\frac{t}{37}}\right)$ what is the percent increase or decrease? _____

Note all numbers are scientific measurements.

A common base for exponentials in many scientific models is e. We can work with this base using calculation devices which will have an e^x button or an $\exp(x)$ function.

Example 6.1.28 What is the percent increase or decrease for the exponential model $P = 27e^{0.0200t}$?

Just as with the fractions we convert to decimal. Note that we can also write the model $P = 27(e^{0.0200})^t$. Do you see the difference? $e^{0.0200} \approx 1.02$ which we found using a device. Thus this is a percent increase of 2%. ▲

Checkpoint 6.1.29 In the model $A = 186.1e^{0.0216t}$ what is the percent increase or decrease? _____

Note all numbers are scientific measurements; round accordingly.

6.1.4 Exercises

Evaluating Exponential Functions. Evaluate these exponential functions.

1. Evaluate the expression $y = 3^x$ for the values given. Round leaving four, non-zero digits for fractional values.

For $x = 4$, $y =$ _____

For $x = -5$, $y =$ _____

2. Evaluate the expression $y = 2.37^x$ for the values given. Round to three significant digits.
For $x = 5$, $y =$ _____
For $x = -5$, $y =$ _____

3. Approximate e^{-4} to 4 decimal places. _____
Approximate $e^{-5.79}$ to 4 decimal places. _____

Reading Exponential Models. Identify the requested properties of these models.

4. In the model $A = 186.1e^{0.0216t}$ what is the percent increase or decrease? _____
Note all numbers are scientific measurements; round accordingly.

5. Consider the model $A = 170.1(1.0113)^t$
What is the initial amount? _____
What is the percent increase or decrease? _____
Note all numbers are scientific measurements; round accordingly.

6. Consider the model $A = 159.2(0.9885)^t$
What is the initial amount? _____
What is the percent increase or decrease? _____
Note all numbers are scientific measurements; round accordingly.

7. Consider the radioactive decay model for Barium-133: $A = 33.9\left(\frac{1}{2}\right)^{t/10.74}$
where t is in units of years.
What is the initial amount? _____
What is the annual percent increase or decrease? _____
Use 365 days/year, 8760 hours per year, and 525600 minutes per year.

8. Consider the compound interest model: $A = 6200\left(1 + \frac{0.0419}{12}\right)^{12t}$
where t is in years and interest is compounded monthly.
What is the initial amount? _____
What is the annual percent yield (annual percent increase)? _____

Exponential Applications. Evaluate exponential functions as needed to answer these questions.

9. Consider the radioactive decay model for Iodine-123: $A = 27.9\left(\frac{1}{2}\right)^{t/13.2}$
where t is in units of hours.
What is the initial amount? _____
What is the amount after 46.2 hours? _____

10. A population numbers 10,000 organisms initially and grows by 12.6% each year.
Suppose P represents population, and t the number of years of growth. An exponential model for the population can be written in the form $P = a \cdot b^t$ where
$a =$ _____ and $b =$ _____.

11. The number of bacteria in a culture is given by the function $n(t) = 955e^{0.15t}$ where t is measured in hours and the result is in grams.
What is the initial mass of the culture (at t=0)? _____
How much bacteria will the culture contain at time t=5? _____

12. Find the final hourly wage if a \$8.00 starting wage is increased by 3.5% each year for 9 years.
The final wage is \$ _____. *Round answer to 2 decimal places*

13. The population of the world in 1987 was 5 billion and the annual growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, find

the projected world population in 1989.

Round your answer to two decimal places.

Your answer is _____ billion

14. A vehicle purchased for $25000 depreciates at a constant rate of 5%. Determine the approximate value of the vehicle 15 years after purchase.

Round to the nearest whole number.

_____

15. A worker's contract states that the hourly wage will start at $9.50 and will increase by $r = 2.5\%$ annually, with the raise given every 12 months.

The hourly wage can be modeled by the exponential formula $S = P\left(1 + \frac{r}{n}\right)^{nt}$, where S is the future value, P is the present value, r is the (nominal) yearly rate of increase, n is the number of times each year that the wage is increased, and t is the time in years.

(A) What values should be used for P, r, and n?

$P =$ _____, $r =$ _____, $n =$ _____

(B) The final hourly wage in 8 years will equal which dollar amount?

Answer = $ _____.

Round answer to the nearest penny.

16. A worker's contract states that the hourly wage will start at $11.50 and will increase by $r = 2.5\%$ annually, with half the annual raise given every 6 months.

The hourly wage can be modeled by the exponential formula $S = P\left(1 + \frac{r}{n}\right)^{nt}$, where S is the future value, P is the present value, r is the (nominal or stated) annual rate of increase, n is the number of times each year that the wage is increased, and t is the time in years.

(A) What values should be used for P, r, and n?

$P =$ _____, $r =$ _____, $n =$ _____

(B) The final hourly wage in 10 years will equal which dollar amount?

Answer = $ _____.

Round answer to the nearest penny.

(C) The actual annual percent increase is _____%.

Round answer to 3 decimal places.

17. In April 1986, a flawed reactor design played a part in the Chernobyl nuclear meltdown. Approximately 14252 becqurels (Bqs), units of radioactivity, were initially released into the environment. Only areas with less than 800 Bqs are considered safe for human habitation. The function $f(x) = 14252(0.5)^{\frac{x}{32}}$ describes the amount, $f(x)$, in becqurels, of a radioactive element remaining in the area x years after 1986.

Find $f(150)$, to one decimal place, in order to determine the amount of becqurels in 2136.

_____

Determine if the area is safe for human habitation in the year 2136.

1. No, because by 2136, the radioactive element remaining in the area is less than 800 Bqs.
2. No, because by 2136, the radioactive element remaining in the area is greater than 800 Bqs.
3. Yes, because by 2136, the radioactive element remaining in the area is less than 800 Bqs.
4. Yes, because by 2136, the radioactive element remaining in the area is greater than 800 Bqs.

18. A newly hatched channel catfish typically weighs about 0.5 grams. During the first 6 weeks of life, its weight increases about 9% each day.

Identify the initial amount:

_____

Identify the growth/decay factor (the b value):

_____

Write an exponential equation to model the situation in the form $f(x) = a(b)^x$

$f(x) =$ _____
How much does the catfish weigh after 4 weeks?
_____ (Round to 2 decimals)

19. Coral reefs throughout the world are dying at a rate of about 3% per year. Write an equation that can be used to determine the future area of a reef that now has an area of 200 km^2. Let c be the area of the coral reef in t years from now.

1. The equation is: _____
2. The equation is in the family:
   (a) quadratic
   (b) linear
   (c) exponential
3. Use the equation to determine how many years until the reef decays to an area of 93 km^2? Round to the nearest year.

   _____

Source

20. Initially, there were 380 wasps in a population, and it decreases by 9% every 7 days.

- What will be the population at $t = 5$ days? [Round your answer to the nearest whole number.]
  Answer: _____
- Write a formula for the population of wasps as a function of t.
  Answer: $P(t) =$_____

21. Atenolol has a half life of 6-7 hours depending on patient factors. For this problem use a half life of 6.9 hours. Suppose Victor is prescribed 50. mg/day. If Victor takes his first 50 mg dose on Monday morning, how much is expected in his blood Tuesday morning at the same time? ____

6.2 Graphs of Exponential Functions

This section addresses the following topics.

- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)
- Identify data varrying directly or indirectly (critical thinking)

We have learned how to identify exponential data (Section 3.4) and have learned to work with some exponential applications (Section 6.1). This section presents the graph (shape) of exponentials and emphasizes two traits of exponential models.

Because devices can quickly produce accurate graphs for us, it is not the goal of this section to help you become proficiant at sketching graphs. Rather, it is the goal for you to be able to read a model, and recognize what different parts of the model imply, because you know what it would look like.

6.2.1 Shape of Exponentials

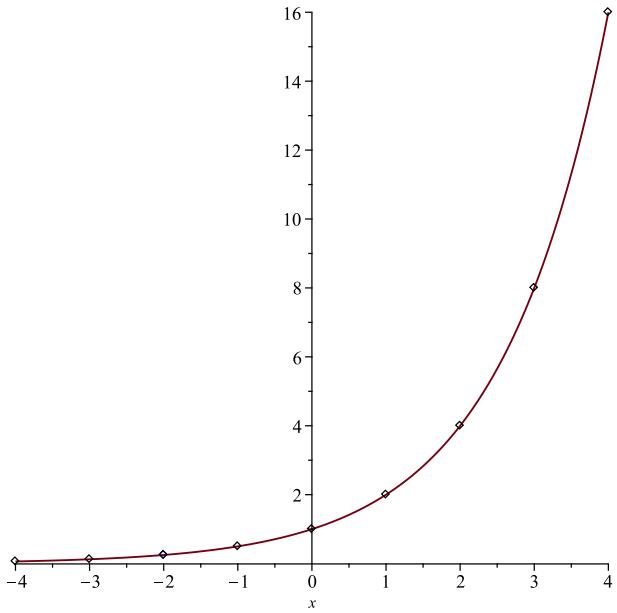
In order to understand the graph of an exponential function we are going to graph one that is not limited by interpretation or rounding.

Example 6.2.1 Graph $y = 2^x$.

First we will generate a table of points with which to start.

x	2^x	
-4	2^{-4}	$= 0.0625$
-3	2^{-3}	$= 0.125$
-2	2^{-2}	$= 0.25$
-1	2^{-1}	$= 0.5$
0	2^0	$= 1$
1	2^1	$= 2$
2	2^2	$= 4$
3	2^3	$= 8$
4	2^4	$= 16$

We can plot these points then sketch a curve smoothly through the points.



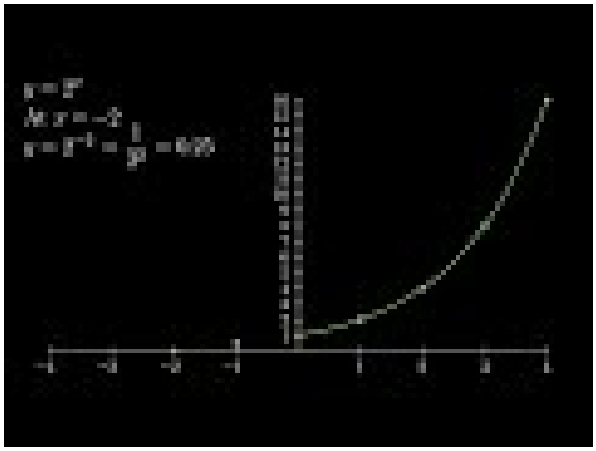
Starting at $x = 0$ consider the curve as it extends to the right. We can use the analogy of walking up hill to understand this curve. At $x = 0$ we are walking up hill so we are a little slower than on a flat surface. At $x = 1$ the height has doubled but so also has the slope (remember the differences are a multiple of the height). Walking the next segment will be more strenuous. At $x = 2$ the height has doubled again, so also has the steepness. Now it is too steep to walk. We will need to scramble or use climbing equipment. Generally an exponential curve becomes increasingly steep.

Because the base of this exponential is 2 (i.e., $y = 2^x$) each time we move one unit to the right the height doubles (times 2). The easiest place to see this is from $x = 0$ to $x = 1$. This is because $y = 2^0 = 1$ and this is true of any base (i.e., $y = a^0 = 1$ for all bases). That means at $x = 1$ the y value will be the base (e.g., $y = 2^1 = 2$).

Starting back at $x = 0$ again consider the curve as it extends to the left. We can see that we are walking downhill; we might pick up a little speed. At $x = -1$ it continues downhill to the left but it is not enough to help us maintain a faster speed. At $x = -2$ we can no longer see the downhill (too gentle) though we would notice it if we were on a bike or skateboard. Just as the curve becomes twice as high and also twice as steep

as we move the to right, it becomes half as high and half as steep as we go to the left. The inability to "see" the downhill nature illustrates that the exponential to the left begins to look like a horizontal line.

Let's clarify that last statement. We notice that all values (left and right) are positive. So to the left the curve is going down, but it never crosses zero. It must therefore become increasingly close to zero. This is why it begins to look like a line namely $x = 0$. When one curve becomes increasingly like another we call that second curve (the line $x = 0$ in this case) an **asymptote**. ▲



Standalone
In Context

This decrease to the left can be understood in contexts as well.

Example 6.2.2 In Checkpoint 6.1.16 we determined that the amount of bacteria was modeled by

$$b = 30(2^{t/25}).$$

Initially ($t = 0$) there are 30 g of bacteria. Now we will calculate how much bacteria there was before this initial measurement.

Time	Mass of Bacteria		
0	$30(2^{0/25})$	$= 30 \cdot 2^{0}$	$= 30$
-25	$30(2^{-25/25})$	$= 30 \cdot 2^{-1}$	$= 15$
-50	$30(2^{-50/25})$	$= 30 \cdot 2^{-2}$	$= 7.5$
-75	$30(2^{-75/25})$	$= 30 \cdot 2^{-3}$	$= 3.75$
-100	$30(2^{-100/25})$	$= 30 \cdot 2^{-4}$	$= 1.875$

We notice that the amount of bacteria decreases. More specifically we see that it is divided in half each time. This is just the reverse of it doubling each time as it grows.

We also notice that because each previous entry is half of the next, there is always some left. No matter how far back in time we go, there will always be some. This is part of all exponentials: rapid growth in one direction and an increasingly gentle decrease toward some amount (zero in this case) in the other direction.

The model measures the bacteria by weight in grams. This can always be cut in half. What reality constraint should we put on this model? ▲

Not all exponential curves increase to the right and decrease to the left. Some reverse (mirror) this pattern. The following example uses radioactive decay as an example of this.

Example 6.2.3 Uranium-242 has a half-life of 16.8 minutes. Suppose we begin with $13\bar{0}$ g of uranium-242.

(a) How much is left at times 16.8, 33.6, and 50.4 minutes?

At time 0 we have $13\bar{0}$ g. At time 16.8 it will be half of the original which is 65.0 g. The 1/2 is an exact number so 3 significant digits is maintained. At time 33.6 it will be reduced by half again ($33.6 = 18.8 \cdot 2$, 2nd half-life). Thus there will be $\frac{65.0}{2} = 32.5$ g. At time 50.4 it will be reduced by half a third time ($50.4 = 16.8 \cdot 3$), so the amount remaining will be $\frac{32.5}{2} = 16.25 \approx 16.3$ g.

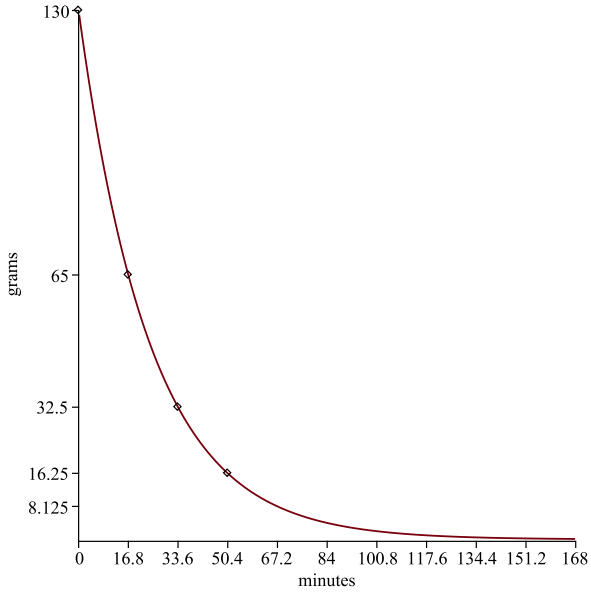
(b) Write a model for this decay.

We can use the process from Subsection 6.1.2. The time in minutes is divided by the half-life 16.8 minutes. Because this is reducing by half the scale is $\frac{1}{2}$.

$$A = 13\bar{0}\left(\frac{1}{2}\right)^{t/16.8}$$

(c) Graph this model.

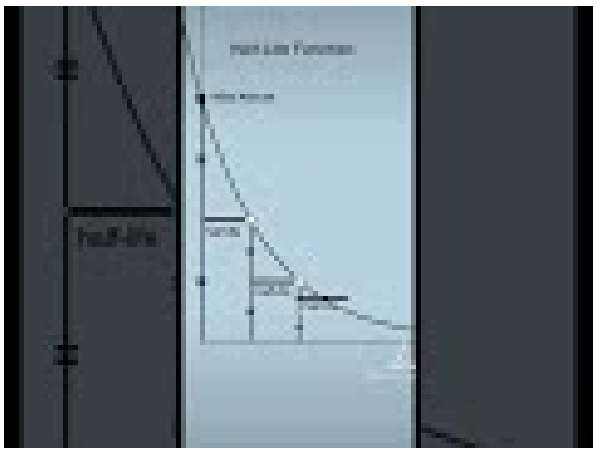
We can use the points from the first task and the method from Example 6.2.1 to produce a graph.



Notice the gentle decrease toward zero is to the right this time as opposed to the left in Example 6.2.1. That is because we are cutting in half each time instead of doubling.

▲

The following video geometrically illustrates the connection between a fixed time (half-life) and decreasing by a ratio (1/2).



Standalone
In Context

6.2.2 Transforming Exponential Functions

In Section 5.4 we learned how to shift and stretch parabolas (graphs of quadratics). Those same transformations work on exponentials (and any other function). The following examples illustrate that the effects are the same.

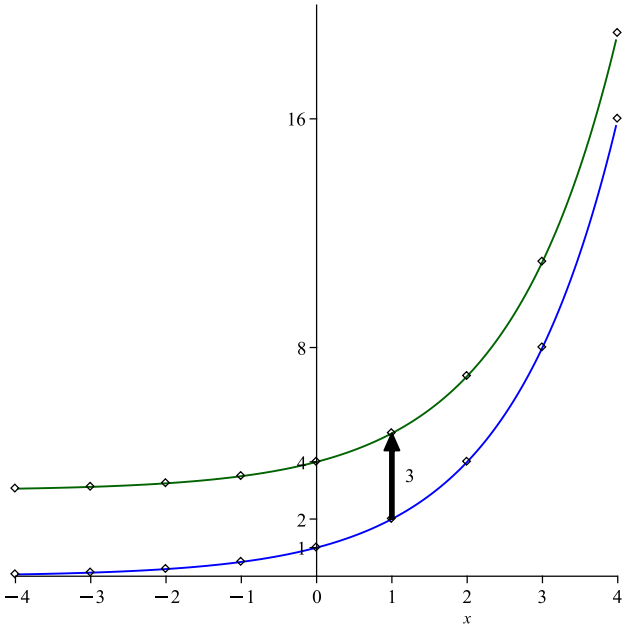
Example 6.2.4 Translate an Exponential. Discover the effect of adding 3 to the fuction by graphing $y = 2^x + 3$ using the graph and table in Example 6.2.1.

We can begin with $x = 1$, from the table we know that $2^1 = 2$, so for this function we end up with $2^1 + 3 = 2 + 3 = 5$. On this function the point (1,2) moved to (1,5). We should check another point. Consider $x = 2$. From the table we know that $2^2 = 4$, so for this function we end up with $2^2 + 3 = 4 + 3 = 7$.

If we look at the rest of the table, we see that every point has the same x value but the y value is 3 units higher.

Graph $y = 2^x + 3$ using the graph and table in Example 6.2.1. We can extend the table to include the +3.

x	2^x	$2^x + 3$
-4	$\frac{1}{16}$	$3 + \frac{1}{16} = 3.0625$
-3	$\frac{1}{8}$	$3 + \frac{1}{8} = 3.125$
-2	$\frac{1}{4}$	$3 + \frac{1}{4} = 3.25$
-1	$\frac{1}{2}$	$3 + \frac{1}{2} = 3.5$
0	1	$1 + 3 = 4$
1	2	$2 + 3 = 5$
2	4	$4 + 3 = 7$
3	8	$8 + 3 = 11$
4	16	$16 + 3 = 19$



Notice that every point is 3 units higher than the point in the original graph. We have moved the graph 3 units up. Because the graph has been shifted up 3 units, the asymptote has also been shifted up three units. It is now $y = 3$.

Because the asymptote was shifted up, we must measure the doubling from one point to the next in terms of height above the asymptote. For example, at $x = 0$ the value is $y = 2^0 + 3 = 4$ so the height above the asymptote is $4 - 3 = 1$. One to the right we have $y = 2^1 + 3 = 5$. The height above the asymptote is $5 - 3 = 2$. From $x = 0$ to $x = 1$ the height above the asymptote doubled as we expect. ▲

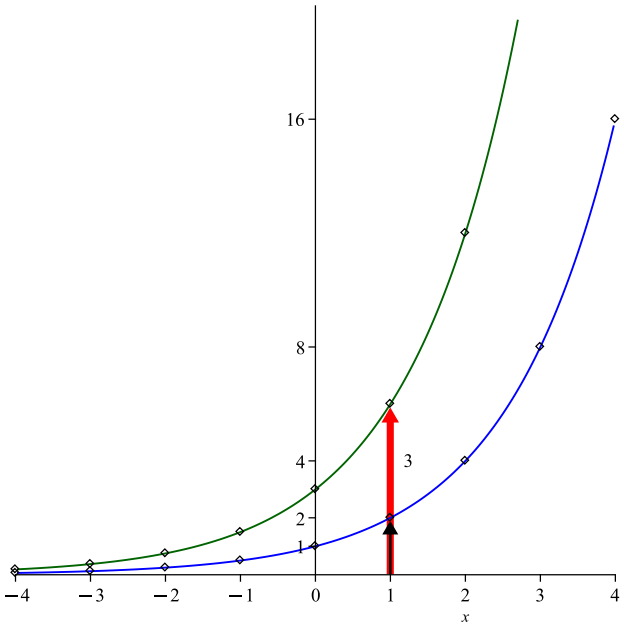
Example 6.2.5 Vertically Stretch an Exponential. Discover the effect of multiplying the fuction by 3 by graphing $y = 3(2^x)$ using the graph and table in Example 6.2.1.

We can begin with $x = 1$, from the table we know that $2^1 = 2$, so for this function we end up with $3(2^1) = 3 \cdot 2 = 6$. On this function the point (1,2) moved to (1,6). We should check another point. Consider $x = 2$. From the table we know that $2^2 = 2$, so for this function we end up with $3(2^2) = 3(4) = 12$. On this

function the point (2,4) moved to (2,12).

If we look at the rest of the table, we see that every point has the same x value but the y value is triple (higher). This is different from adding 3 (bigger growth).

x	2^x	$3(2^x)$
-4	0.0625	$3 \cdot 0.0625 =$ 0.1875
-3	0.125	$3 \cdot 0.125 =$ 0.375
-2	0.25	$3 \cdot 0.25 =$ 0.75
-1	0.5	$3 \cdot 0.5 =$ 1.5
0	1	$3 \cdot 1 =$ 3
1	2	$3 \cdot 2 =$ 6
2	4	$3 \cdot 4 =$ 12
3	8	$3 \cdot 8 =$ 24
4	16	$3 \cdot 16 =$ 48



Notice that every point is 3 times farther from the x-axis. This makes it 3 times steeper. ▲

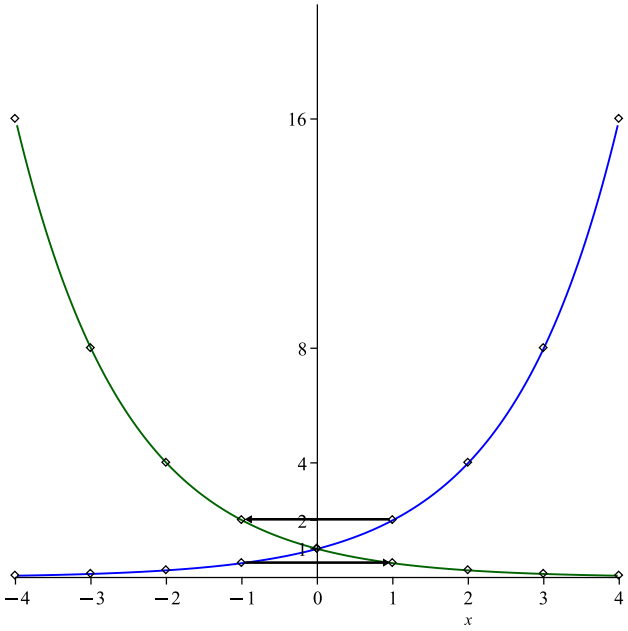
Example 6.2.6 Reflect an Exponential. Discover the effect of negating x by graphing $y = 2^{-x}$ using the graph and table in Example 6.2.1.

We can begin with $x = 1$, so $-x = -1$. This means that when we plug $x = 1$ into this function (2^{-x}) we get the same result (y value) as plugging $x = -1$ into the function 2^x. We can look that up in the table (or simply calculate) $y = 2^{-1} = 0.5$. On this function the point (-1,0.5) moved to (1,0.5).

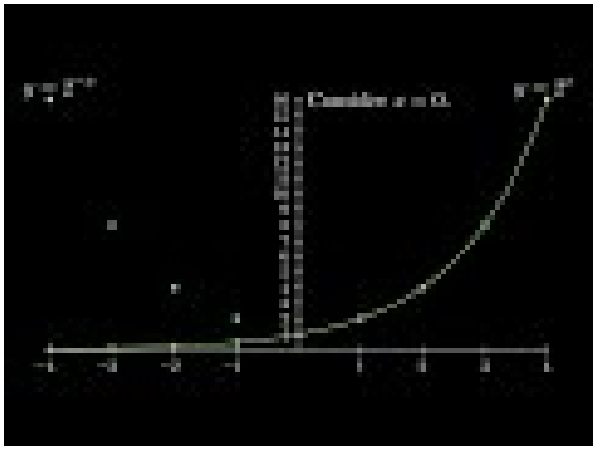
Similarly for $x = -1$, so $-x = -(-1) = 1$. This means that when we plug $x = -1$ into this function (2^{-x}) we get the same result (y value) as plugging $x = 1$ into the function 2^x. We can look that up in the table (or simply calculate) $y = 2^1 = 2$. On this function the point (1,2) moved to (-1,2).

Comparing the results of these, two calculations we see that the points (1,2) and (-1,0.5) swapped position. We can try one more value to see if this continues. Consider $x = 4$. $-x = -4$. Thus we obtain the same result as plugging $x = -4$ into 2^x. This gives us the point (4,0.0625) which used to be at (-4,0.0625). Again a point swapped sides. You can confirm this works on other points by completing the table below.

x	$-x$	2^{-x}	
-4			
-3			
-2			
-1	1	$2^1 =$	2
0	-0	$2^0 =$	1
1	-1	$2^{-1} =$	0.5
2			
3			
4	$-4 =$	$2^{-4} =$	0.0625



Notice that every point ends up on the other side of the y-axis. This is a reflection of the original curve. ▲



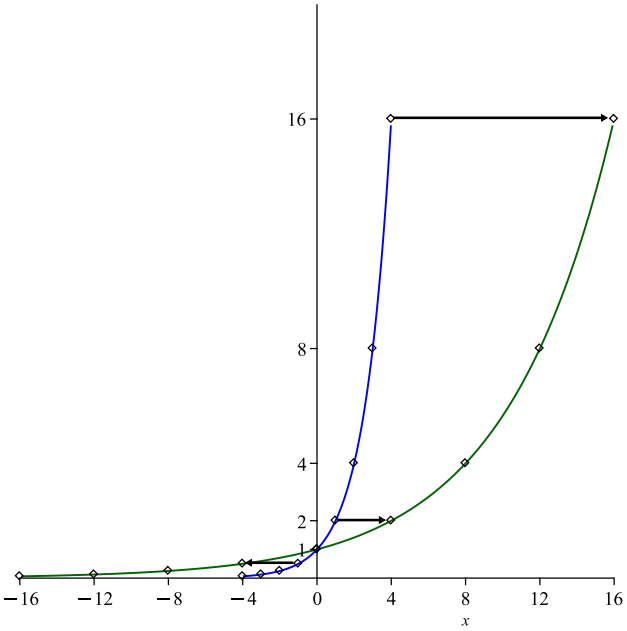
Standalone
In Context

Example 6.2.7 Horizontally Stretch an Exponential. Discover the effect of multiplying x by a number (0.25 in this case) by graphing $y = 2^{0.25x}$ using the graph and table in Example 6.2.1.

We recognize that $0.25 = \frac{1}{4}$, so for our first value we choose $x = 4$. We calculate $0.25(4) = 1$. This means that when we plug $x = 4$ into this function ($2^{0.25x}$) we get the same result (y value) as plugging in $x = 1$ to the function 2^x. We can look that up in the table (or simply calculate) $y = 2^{0.25(4)} = 2^1 = 2$. On this function the point (1,2) moved to (4,2).

To understand the full effect, complete the table below using the same process. Because multiplying by 0.25 changes the x value, we have changed the x values in this table so that all points will match points in the table we are using.

x	$0.25x$	$2^{0.25x}$
-16		
-12		
-8		
-4	$0.25(-4) = -1$	$2^{-1} = 0.5$
0	$0.25(0) = 0$	$2^0 = 1$
4	$0.25(4) = 1$	$2^1 = 2$
8		
12		
16	$0.25(16) = 4$	$2^4 = 16$



The graph shows that we end up with the same y values, but they occur spread out over a wider area. Multiplying 0.25 by x stretched the curve horizontally. Because the number is between 0 and 1 (small fraction) it stretched it out. We could squeeze it in by multiplying by a number larger than one (e.g., 2^{4x}). ▲

Now that we know how each piece affects the result we can put them all together. Our general equation for a base of 2 is

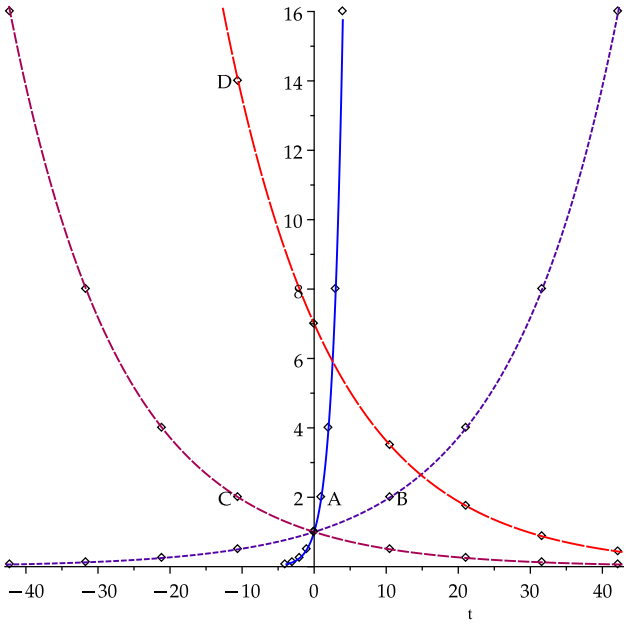
$$y = A \cdot 2^{B(x-h)} + k$$

where h and k represent the horizontal and vertical shifts respectively. A and B represent the vertical and horizontal stretches respectively. The first two examples below start by using each of these transformations individually. The third example illustrates using a shorter method that we typically use.

Example 6.2.8 Graph the model $7.000 \cdot 2^{-t/10.551}$ from Example 6.1.14. In order to match our examples above we can express the exponent as $7.000 \cdot 2^{-0.094778t}$. We simply calculated 1/10.551 and maintained 5 significant digits.

First, we identify the changes from $y = 2^t$. The coefficient of the exponent is a horizontal stretch. The negative in the exponent is a reflection over the y-axis. The 7.000 in front is a vertical stretch. We can make these changes in order of operation: horizontal stretch, reflection, vertical stretch. These changes are shown one at a time in the figure below. Graph A is the original. B shows the horizontal stretch. C shows the

reflection. D adds the vertical stretch. The labels are located to show how one point is transformed by each step.

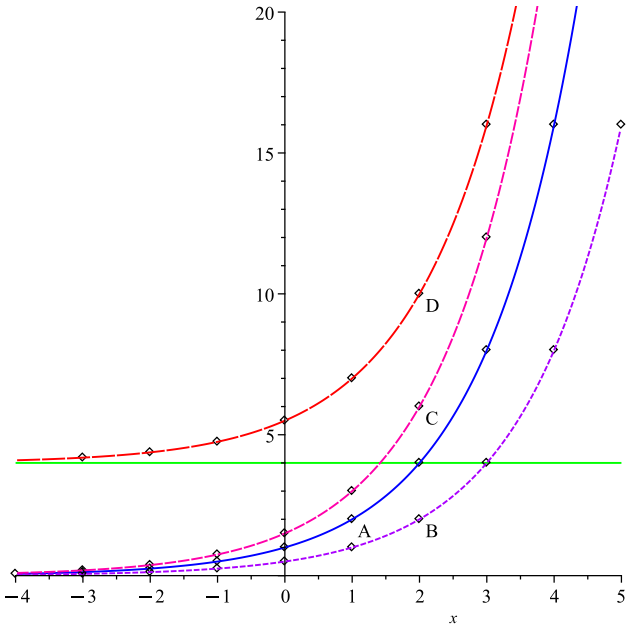


Looking at the final graph we can note that we really needed only three details to graph it. These are the asymptote (which remained unchanged from $y = 0$ here) and two points. One point tells us where to start and the other indicates how steep (because of the change between the two points). ▲

Before demonstrating graphing using the asymptote and two points, we graph an exponential with shifts as well.

Example 6.2.9 Graph the equation $y = 3\left(2^{x-1}\right) + 4$.

First, we identify that this is a modified version of $y = 2^x$. Next we identify the changes. The -1 is a horizontal shift. The 3 is a vertical stretch. The +4 is a vertical shift. We perform the changes in this order (following order of operations). These are shown in the graph below. Graph A is the original. B shows the horizontal shift. C shows the vertical stretch. D adds the vertical shift. The labels are located to show how one point is transformed by each step. Notice that the asymptote was moved up 4 units by the vertical shift as well.



Again, we could have sketched the asymptote knowing only the vertical shift would change it. Then we could plot points at $x = 0, 1$. The curve is then sketched down to the left (to approach the asymptote) and up to the right with steepness determined by the second point. ▲

Now that we have practiced using the various transformations, we can look at how a graph can be produced from just two, particular points. First, we consider that $2^0 = 3^0 = e^0 = 1$; no matter the base the zeroth power is 1. Thus the first of the two points has the x coordinate such that the base is raised to the zero power. Second, we know that the asymptote is one unit below this point. Once we locate this first point, we can sketch the asymptote.

Third, we note that $2^1 = 2$, $3^1 = 3$, $e^1 = e$. Thus the base can be determined by looking at the point with x coordinate one unit left/right from the first point. In the following example we use these three ideas to identify the shifts and reflection.

Example 6.2.10 Graph the equation $y = 3^{x+2} + 5$.

First, we calculate the x value where $3^{x+2} = 3^0$. This is where $x + 2 = 0$ or $x = -2$. This implies that the curve is shifted to the left (we start graphing two left of zero instead of at zero). At $x = -2$ we calculate $y = 6$. We can plot this point.

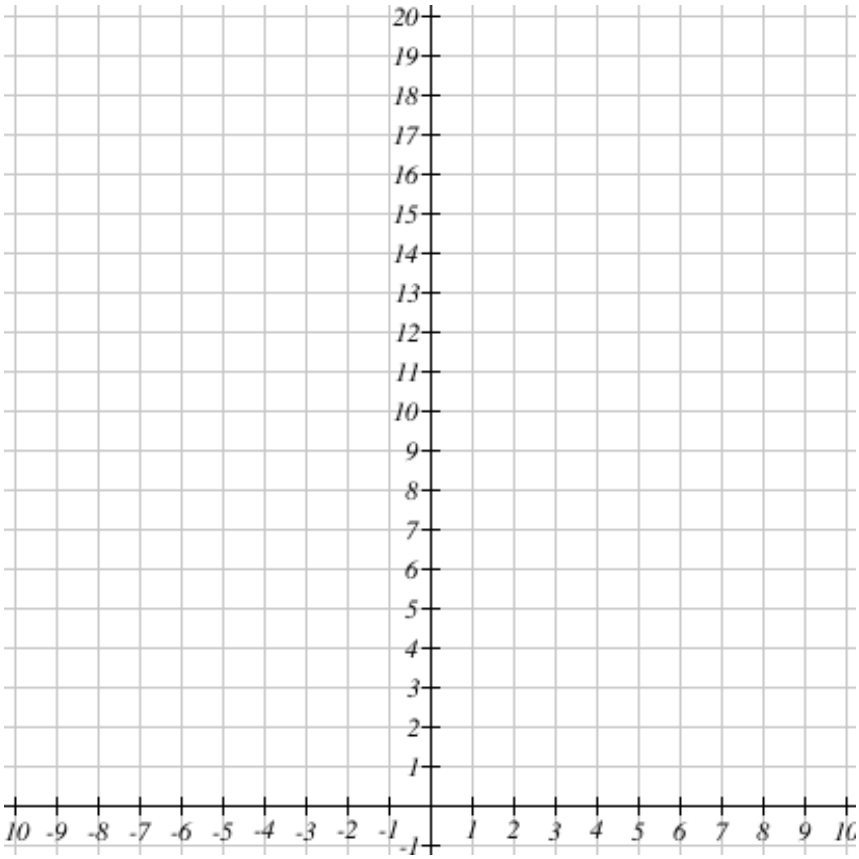
Because the asymptote is always one unit lower than this point, we now know the asymptote is $y = 5$.

Because this is an increasing exponential (positive exponent), the next point we want is one unit to the right. $x = -2 + 1 = -1$. At this x value we calculate $y = 8$. Because we know the asymptote is at $y = 5$, we know this point is $8 - 5 = 3$ units above the asymptote. This matches the base of 3 (i.e., the 3 in $3^{x+2} + 5$).

So from these two points, we can determine the asymptote, the base, and the vertical shift. This is enough for graphing. When you graph exponentials in exercises you will enter these two points, and the software will calculate the rest using these ideas. ▲

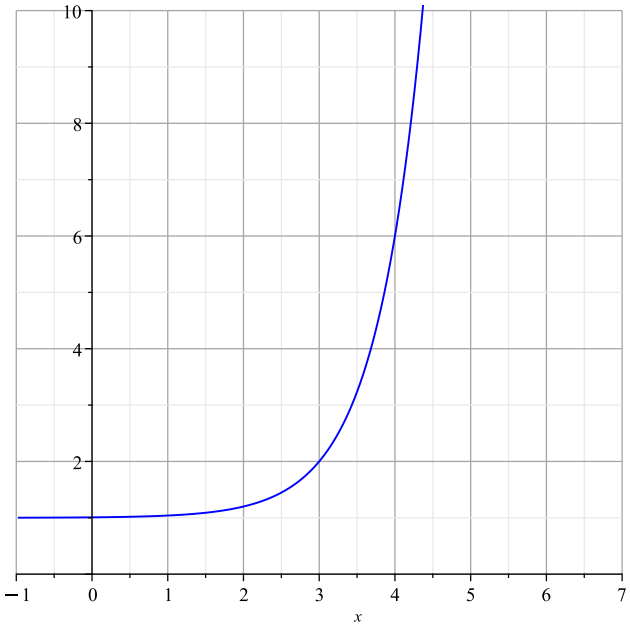
Similar ideas enable us to use just two points if we have horizontal or vertical stretches, but the explanation is more complicated than is useful in this text. We should be aware that a vertical stretch means the asymptote is not one unit beneath the first point ($x = -2$ in the example above), because the stretch, stretches that distance. For example for $3(2^{x-3})$ the asymptote is at $y = 0$, the first, convenient point is at $x = 3$ which gives $y = 3$. This is 3 above the asmyptote instead of just 1.

Checkpoint 6.2.11 Graph the equation $y = 3\left(4^{-x+2}\right) + 1$. You will select a point on the asymptote (only y coordinate matters); then you select two points on the exponential.



The following examples use these ideas in reverse as you will be expected to do. We read the base, vertical shift, and horizontal shift from a graph.

Example 6.2.12 The graph below is of the equation $y = b^{x-h} + k$. Determine the values of b, h, and k.



First, by looking at the left side of the graph we notice that the graph is always above $y = 1$. This must be the asymptote. The asymptote gives us the vertical shift, i.e., $k = 1$.

We know all exponential graphs have a point one unit higher ($y = 2$ in this case). This point occurs at

$x = 3$. Because this point on an unshifted exponential is at $x = 0$, the horizontal shift $h = 3$.

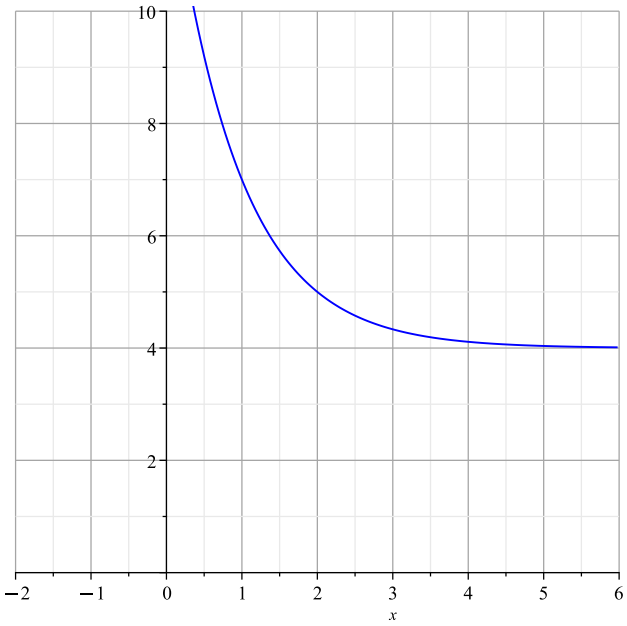
Finally, we can look one unit to the right of this point ($x = 4$). The y value there is $y = 6$. This is $6 - 1 = 5$ units above the asymptote. This is the base $b = 5$.

The equation is

$$y = 5^{x-3} + 1.$$

▲

Example 6.2.13 The graph below is of the equation $y = b^{\pm(x-h)} + k$. Determine the values of b, the sign, h, and k.



First, we notice that the graph decreases to the right instead of the left. We know this is a reflection so it will be $-x$.

Second, because it is decreasing we know the asymptote will be on the right. All the points are above $y = 4$ thus the asymptote is $y = 4$. The asymptote gives us the vertical shift, i.e., $k = 4$.

We know there will be a point one unit higher ($y = 5$ in this case). This point is at $x = 2$. This tell us the horizontal shift is $h = 2$.

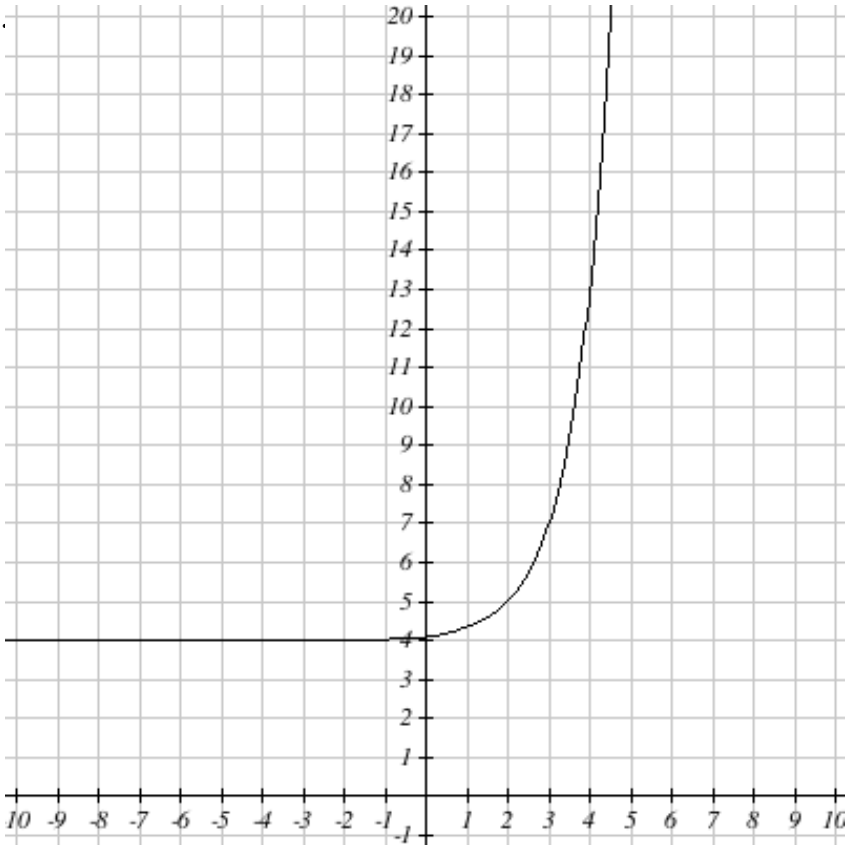
Finally, we can look one unit to the left of this point ($x = 1$). The y value there is $y = 7$. This is $7 - 4 = 3$ units above the asymptote. This is the base $b = 3$.

The equation is

$$y = 3^{-(x-2)} + 4.$$

▲

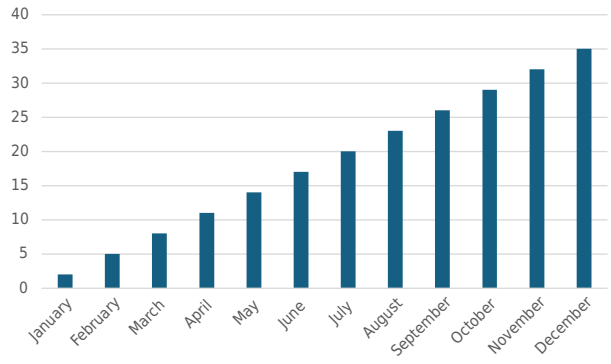
Checkpoint 6.2.



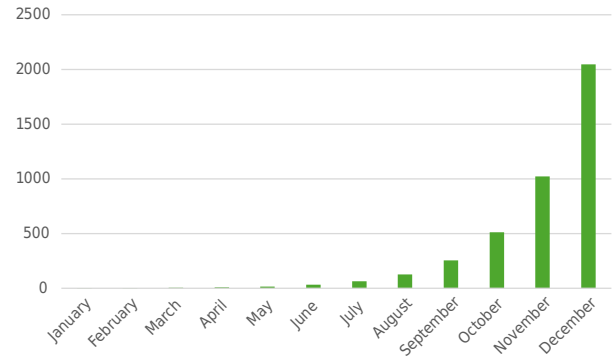
Write the equation for this graph. It is in the form $y = b^{\pm(x-h)} + k$.
$y =$_____

6.2.3 Identifying Exponential Graphs

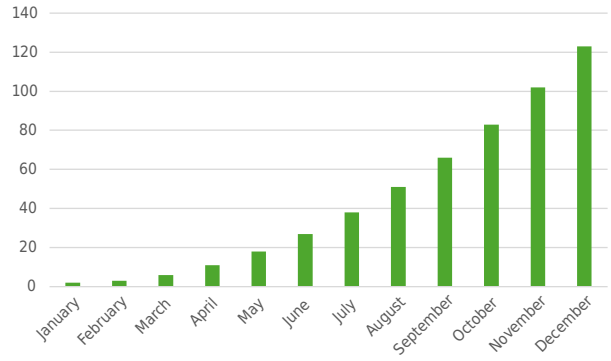
When we look at data, which may not be labeled as linear, quadratic, or exponential, how can we quickly determine which one it is? Look at the four examples below and try to determine which growth rate each has. Note one graph is a growth rate we have not covered.



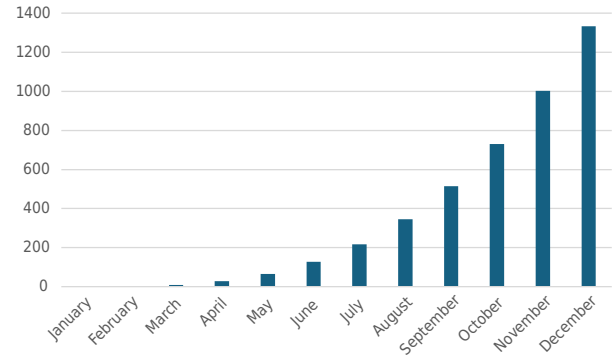
(a) Bar Chart 1



(b) Bar Chart 2



(c) Bar Chart 3



(d) Bar Chart 4

Figure 6.2.15 Identify the Rates of these Charts

A linear relationship is defined by the differences between pairs of data points being the same. On bar graphs this means the difference in height between consecutive bars must be the same. If we compare the change from January to February to the change from November to December we see that Figure 6.2.15(b)–6.2.15(d) they are vastly different (very small early and much larger later). For Figure 6.2.15(a) however, the increase between bar heights looks the same. If we wished, we could use a ruler to measure and convince ourselves. This appears to be linear.

An exponential relationship can be recognized by the ratio between pairs of data points being the same. This is hard to use on a bar graph, because it means the height of each bar is a multiple of the previous. For example each bar would be 2 or 3 times as high. This would be hard to identify if each bar is 2.3 times as high however. Rather we will use two, alternative traits.

We know that the graph of an exponential is nearly flat in one direction and very steep in the other. Figure 6.2.15(c) does not appear to flatten heading left. The bottom two graphs do have this flat appearance to the left. The other trait is that the differences of consecutive terms in an exponential relationship are the same scale as the entries. This means the change in height of consecutive bars should be on scale with the size of a bar. Consider September and October on Figure 6.2.15(b). The change in height from September to October looks like the height of the September bar. In contrast the change in height from September to October in Figure 6.2.15(d) is much less than the height of the September bar. Thus we can be convinced that the data in Figure 6.2.15(b) is exponential and none of the others are.

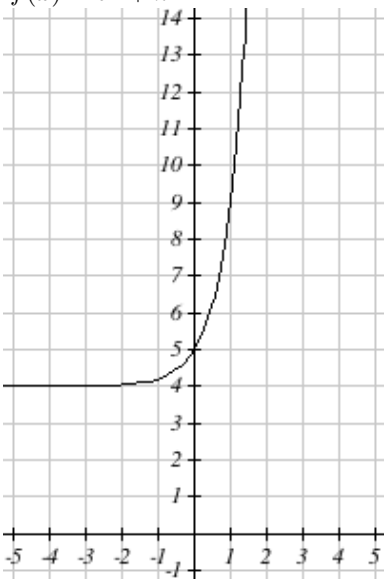
We identify quadratic data by looking at the second differences. To do this on a bar graph would mean constructing a bar graph where each new bar's height is the change in height of consecutive bars on the original graph. Then we would look if this new bar graph was linear. This is not a reasonable task. We would be better off asking for the original data.

The two bar graphs on the right have different relationships: one is quadratic, the other is not. Both can be described in the same terms (e.g., change in heights is increasing). This is a warning to not make many assumptions about data from limited data or simple graphs.

6.2.4 Exercises

Identify the Exponential Function. Use the information provided to write an equation for each exponential function.

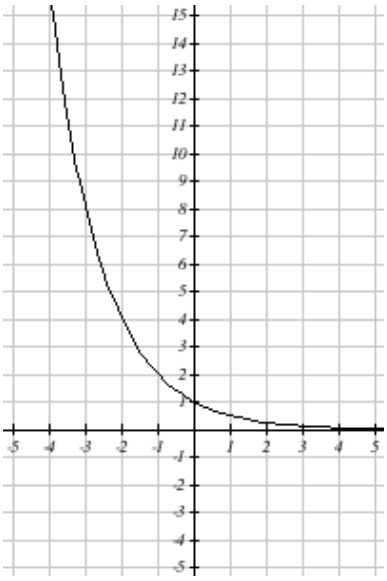
1. The function below has the form $f(x) = b^x + k$.



Which of the following functions is shown on the graph?

1. $f(x) = 5^x + 5$
2. $f(x) = 5^x + 3$
3. $f(x) = \left(\frac{1}{5}\right)^x + 3$
4. $f(x) = \left(\frac{1}{5}\right)^x - 4$
5. $f(x) = \left(\frac{1}{5}\right)^x + 5$
6. $f(x) = \left(\frac{1}{5}\right)^x + 4$
7. $f(x) = 5^x + 4$
8. $f(x) = 5^x - 4$

2. What function is graphed below?



1. $f(x) = 4^x$
2. $f(x) = \left(\frac{1}{3}\right)^x$
3. $f(x) = 2^x$
4. $f(x) = \left(\frac{1}{2}\right)^x$
5. $f(x) = 3^x$
6. $f(x) = \left(\frac{1}{5}\right)^x$
7. $f(x) = 5^x$
8. $f(x) = \left(\frac{1}{4}\right)^x$

3. The table below shows values from the function $f(x) = b^x$. Identify b.

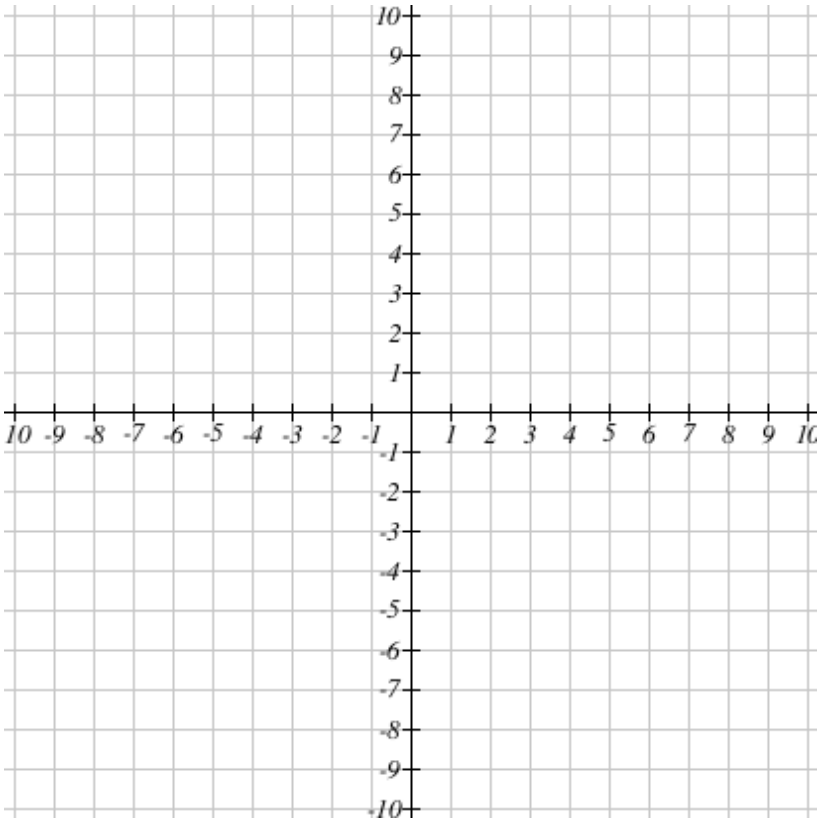
x	-2	-1	0	1	2
$f(x)$	81	9	1	$\frac{1}{9}$	$\frac{1}{81}$

$b =$ _____

4. Find the exponential function $f(x) = Ca^x$ whose graph goes through the points $(0, 5)$ and $(2, 20)$.
$a =$ _____ ,
$C =$ _____ .

5. Find a formula for the exponential function passing through the points (-1,$\frac{2}{3}$) and (1,6)
y = _____

6. Graph the Exponential. Draw the graph of $f(x) = 6^{-x+4}$



Distinguish Growth and Decay. For each problem determine if it is a case of exponential growth or decay.

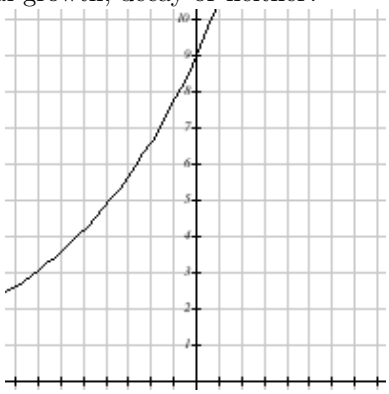
7. Determine whether the following equation represents an exponential growth or exponential decay.
$y = 228 \cdot (1.97)^x$

1. exponential decay
2. exponential growth

8. Determine whether the following equation represents an exponential growth or exponential decay.
$y = 33.6 \cdot (0.73)^x$

1. exponential decay
2. exponential growth

9. Does this graph show exponential growth, decay or neither?

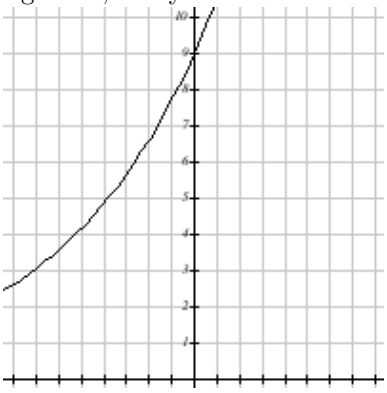


1. Growth

2. Decay

3. Neither

10. Does this graph show exponential growth, decay or neither?



1. Growth

2. Decay

3. Neither

11. Does this graph show exponential growth, decay or neither?
$3(4)^x$

1. Growth

2. Decay

3. Neither

12. Does this graph show exponential growth, decay or neither?
$3(4)^x$

1. Growth

2. Decay

3. Neither

6.3 Logarithm Properties

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Read and interpret models (critical thinking)

The previous two sections presented exponential models, and we answered some questions about applications using these models. In order to solve more problems involving exponential models we need the **logarithm** which is presented here. This section presents a definition of the logarithm as well as presenting how to graph them and one application. These latter will help us understand how it works.

Historically logarithms also have a connection to drunk pigeons and a Scotsman's bones.

6.3.1 Definition of Logarithm

The following definition is commonly used when we are using logarithms to solve equations.

Definition 6.3.1 Logarithm. $\log_a(x) = y$ if and only if $a^y = x$ ◆

This definition is simply that the logarithm is the opposite of an exponential in the same sort of way that a square root is the opposite of a square.

Example 6.3.2 Each of the following is a conversion between exponential and logarithmic notation.

(a) $\log_2(8) = 3$ is the same as $2^3 = 8$.

(b) $\log_5(1/25) = -2$ is the same as $5^{-2} = 1/25$.

(c) $\log_{10}(\sqrt{10}) = 1/2$ is the same as $10^{1/2} = \sqrt{10}$.

▲

Checkpoint 6.3.3 Write $\log_5(625) = 4$ in exponential notation. _____

Use a calculator to confirm that the statement is true.

Checkpoint 6.3.4 Write $36^{\frac{5}{2}} = 7776$ in logarithmic notation. _____

6.3.2 Graphing Logarithms

The purposes of this section are to become proficient with the defition through practice, and to understand the logarithm through practice and seeing its graph.

Example 6.3.5 We will graph $y = \log_2(x)$. As before we will begin by completing a table. Before we can fill out a table we need to figure out how we can produce points.

While we would normally pick values like $x = 3$ plug that in, producing $y = \log_2(3)$, and calculating, that will not work with logs. Indeed we will never calculate this particular point. Because it is easier to evaluate exponentials than logs, we will fill out a table of related exponentials then converting the points. The related exponential here is $2^y = x$.

y	$x = 2^y$
0	$2^0 =$ 1
1	$2^1 =$ 2
2	$2^2 =$ 4
3	$2^3 =$ 8
4	$2^4 =$ 16

To produce points on the graph of this logarithm, we need only note that x and y reverse by our definition. The next table was constructed by swapping the x and y columns of the previous table.

x	$y = \log_2(x)$
1	0
2	1
4	2
8	3
16	4

Notice that as promised we do not have a point with $x = 3$. This is because our method works only on values of x that are powers of 2.

To obtain points left of $(1, 0)$ we will need to expand the table of points for the exponential. The previous table used positive y values. So we will select negative y values this time.

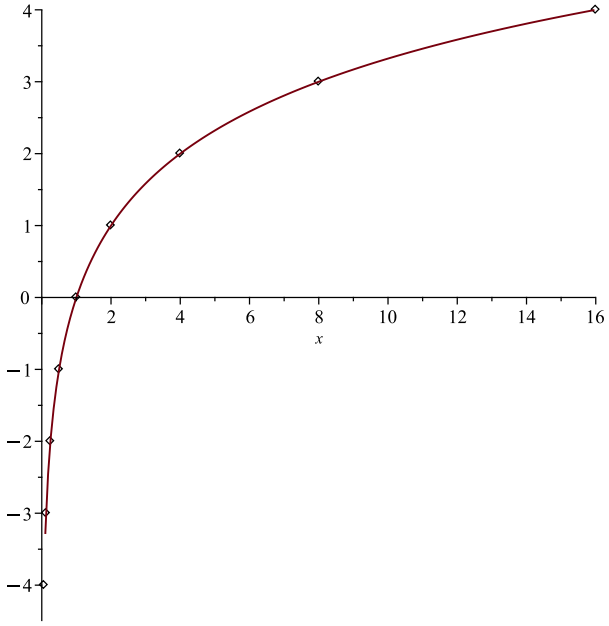
y	$x = 2^y$
-1	$2^{-1} = 1/2$
-2	$2^{-2} = 1/4$
-3	$2^{-3} = 1/8$
-4	$2^{-4} = 1/16$

Again we produce points on the graph of the logarithm by swapping the x and y coordinates.

x	$y = \log_2(x)$
$1/2$	-1
$1/4$	-2
$1/8$	-3
$1/16$	-4

These are points to the left of the previous ones. However, there are none with negative x values. To understand why consider what a negative x value implies. Consider $\log_2(-4) = y$ means $2^y = -4$. However, now matter how many times we multiply positive 2 it will never be negative. Therefore negative x values do not make sense in a logarithm.

The graph of $y = \log_2(x)$ based on these points is below.



▲

Checkpoint 6.3.6 Following the previous example construct a table for $x = 3^y$, then use it to construct a table for $y = \log_3(x)$. Finally, graph this logarithm.

Next we consider the shape of the graph and what it tells us about logarithms. The first question is how high the logarithm curve grows. To explain this question consider Figure 6.3.7 below. As we look from left to right both curves are always increasing. Both curves increase by less as we go to the right. However, the lower (blue, dashed) curve levels off. That is it never increases above $y = 1$. We want to know if the logarithm curve ever levels off.

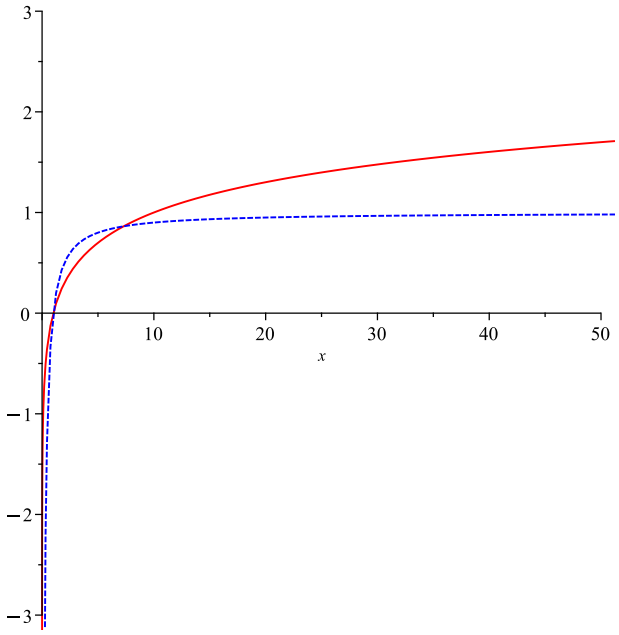


Figure 6.3.7 Graph of two always increasing curves

Example 6.3.8 Consider the example $y = \log_2(x)$. We know from the table above that it grows above $y = 1$. Consider whether it grows above $y = 100$. This would mean $100 < 2^x$. We know it is easier to calculate this logarithm for powers of 2 so instead we consider

$$100 < 128 = 2^7.$$

Does it grow above $y = 1000$. We can use

$$1000 < 1024 = 2^{10}.$$

By now we can see a pattern. For any y value we select there is a power of 2 that is greater (we can double forever). The logarithm evaluated at that power is bigger. Thus we know the logarithm increases forever. This is not dependent on the base 2. The same statement is true of any number (e.g., 3^x and 29^x increase forever as well).

In practice this means we should draw the graph of any logarithm with the curve pointing slightly up at the end. ▲

We claimed above that the two graphs in Figure 6.3.7 increase more slowly as they continue to the right. The next example explains why this is true of the graph of a logarithm.

Example 6.3.9 Looking at the table of points for $y = \log_2(x)$ in Example 6.3.5 we note that the curve increased by 1 in height from $x = 1$ to $x = 2$. The next increase of 1 in height is at $x = 4$. The next one is at $x = 8$. This is a jump of 1 then 2 then 4. Each time the distance we have to travel to see an increase in height of 1 unit doubles. This means the rate of increase was $1, 1/2, 1/4$. This is a decreasing sequences, so while the logarithm increases forever, it does so increasingly slowly. ▲

The final trait of the graph of a logarithm to consider is what happens as the graph goes to the left. We have already established that it is not defined for negative x values. However, the graph does not have a starting point (first point on the left). The next example illustrates the asymptotic nature of the left part of the graph.

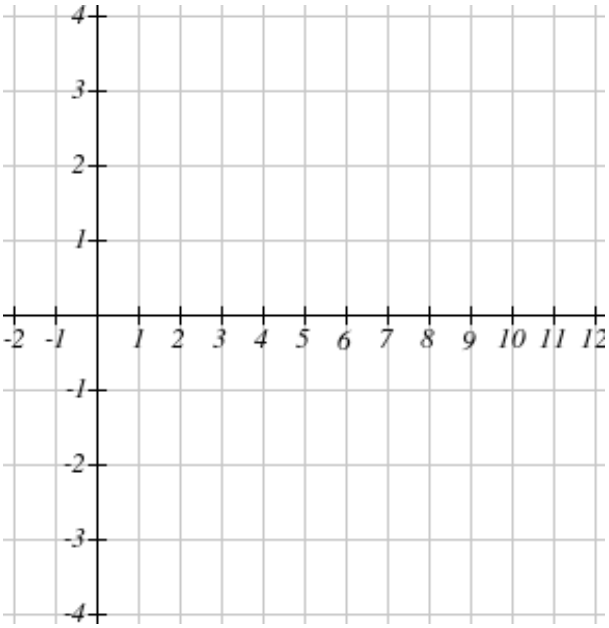
Example 6.3.10 First consider the question, is there an x value such that $0 = 2^x$? We know that $2^0 = 1$. If we plug in negative values we obtain fractions, e.g., $2^{-3} = 1/2^3 = 1/8$. If we use a bigger negative number (e.g., -29) we end up with a smaller fraction ($2^{-29} = 1/2^{29}$). No matter how far we go, 0 is never the result. Because of this $\log_2(0)$ is not defined.

The calculations above also illustrated that we can get as close to zero as we want (just pick a big enough

negative number). This means that just as the exponential curve has an asymptote to the left (approaches $y = 0$), so the logarithm curve has an asymptote to the left, namely $x = 0$. ▲

Putting these traits together we realize the graph of any logarithm can be determined by the vertical asymptote and two points. This matches exponential graphs. From the left most point we draw asymptotically toward the vertical asymptote. To the right we draw with a slight increase.

Checkpoint 6.3.11 Graph $y = \log_8(x)$ by first placing the vertical asymptote.



6.3.3 Special Logarithms

Some logarithms occur sufficiently frequently in applications that they have their own notation.

Logarithms were developed to manage large numbers base 10. Thus we call base 10 logarithms: **common logs**. This is written without the base. For example $\log(100) = 2$ is the same as $\log_{10}(100) = 2$.

In science we often use another log called the **natural logarithm**. This is written as $\ln(x)$ (for logarithm natural). Natural logarithms are paired with the base e, so $\ln(x) = \log_e(x)$. As a result $\ln(5) = y$ is the same as $e^y = 5$. To calculate natural logs and values of e^x you will use technology: knowing a value for e is unnecessary. For the curious $e \approx 2.1718281828$.

Checkpoint 6.3.12 Evaluate the following logarithms using a device. Round to 3 decimal places.

$\log\left(1.22 \times 10^{20}\right) =$ _____

$\ln(5.76) =$ _____

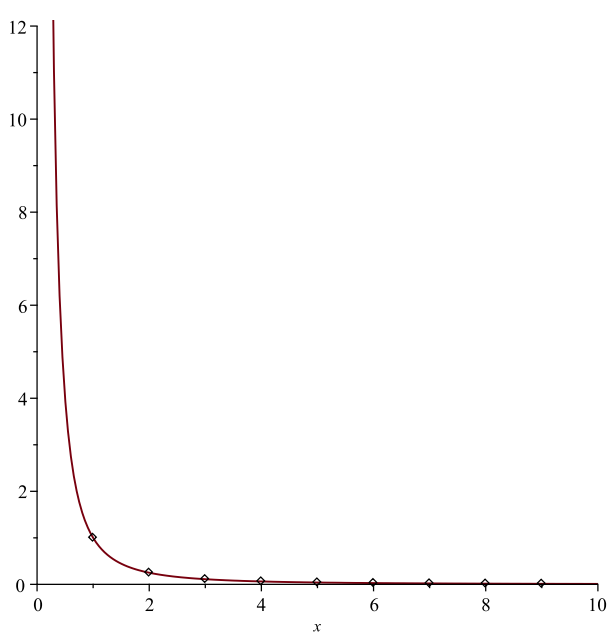
6.3.4 Logarithmic Scale

This section demonstrates how logarithms can be used to better analyze data. The technique is to modify how we graph, specifically how we label one axis.

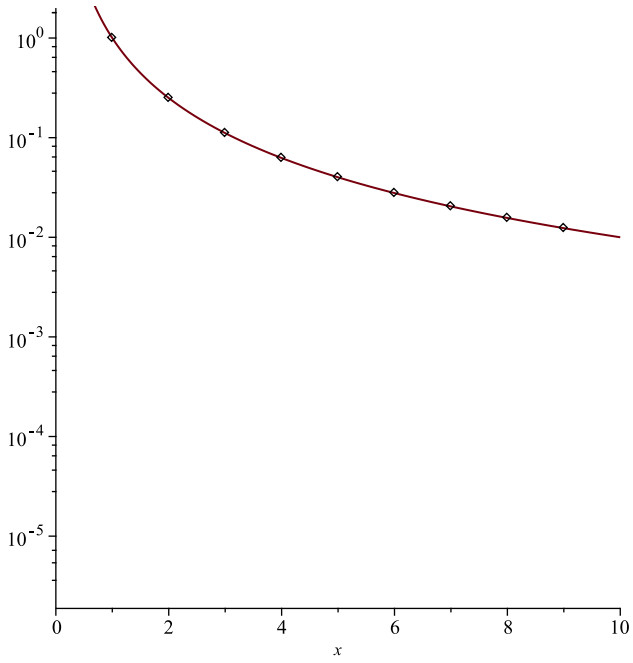
Often the scales on the two axes are different. Consider the graph of $y = \log_2(x)$ in Example 6.3.5 above. The x-axis is labeled 0 to 16 and the y-axis is labeled - 4 to 4. That means the x-axis, though it takes up the same length of area as the y-axis represents 16 units to the y-axis' 8 units. We do this because many graphs would otherwise be too tall or too wide to fit.

What these graphs do require is that each tick mark represents the same number of units. For example on the graph of this log, every tick mark on the x-axis represents one unit (only half are labeled). On the y-axis every tick mark represents 1/2 unit (only half are labeled). Sometimes this requirement means it is hard to read a graph.

Consider the two graphs in Figure 6.3.13. Note they are the same data drawn with different scales. Write down coordinates for the first three points. Is it easier to be precise using the version on the left (normal scale) or the version on the right (log scale)?



(a) Normal Scale



(b) Log Scale

Figure 6.3.13 Comparison of Scales

The graph on the left uses the normal (linear) scale. Specifically on both the x and y axes each tick represents the same difference (add 1 unit for each tick or 2 units for each label). The graph on the right is the same data using a logarithmic scale. The x axis is still linear (same as the left), but the y axes is labeled such that each tick is the same product (multiply by 10 for each label). Basing a scale on multiplying rather than adding makes this match exponentials. It is called a log scale because it looks like graphing $(x, \log(y))$.

When is this scale useful? Any time data is growing by a multiple rather than by an amount added. This includes anything exponential. Recall that exponential growth means growing by a fixed percent. Signal strength in electronics is an example. Consider how much of the signal is lost as it passes through a long cable. The resistance of the cable will cause a fixed percent to be lost rather than a fixed amount. A stronger signal will lose more (but still be stronger).

Consider an electronic amplifier: a device to increase intensity of sound by increasing the power of the signal. The mechanism used multiplies the incoming signal power by some factor. This enables your media devices to send out a very weak audio signal, but the speakers to receive a lot more power which they need to move enough air to be heard. Based on this design if more than one amplifier is connected in series, each one multiplies the signal strength from the previous one.

Studies of the human brain indicate that our brains interpret sensory data using a logarithmic scale. That is, we perceive things based on how much of a multiple they are of current condition rather than how much is added. For example, a 10 degree temperature change feels like more of change if the initial temperature is 20° F (when it is half of the current value) than if it is 50° F (now only one fifth of the current value).

6.3.5 Exercises

Evaluate the Logarithms. Evaluate the logarithm using the definition or a device as directed.

1. Write the following logarithmic equation in exponential form
$\log_9(9) = 1$

_____

2. Write the following logarithmic equation in exponential form
$\log_9(9) = 1$

_____

3. Evaluate the following expressions.
(a) $\log_5 5^9$ = _____
(b) $\log_3 81$ = _____
(c) $\log_4 1024$ = _____
(d) $\log_5 5^5$ = _____

4. Find the logarithm.
$\log_3\left(\frac{1}{27}\right)$=_____

5. Evaluate using your calculator, giving at least 3 decimal places:
$\log(530)$ = _____

6. Evaluate $\ln(0.119)$. Give your answer to 4 decimal places.
$\ln(0.119)$ = _____

7. Write the following exponential equation in logarithmic form
$9^{-1} = \frac{1}{9}$

_____

8. Write the following exponential equation in logarithmic form
$9^{-1} = \frac{1}{9}$

_____

9. Express each equation in logarithmic form.
(a) $e^x = 8$ is equivalent to the logarithmic equation: _____
(b) $e^4 = x$ is equivalent to the logarithmic equation: _____

Solve Logarithms. Use the definition of logarithm to solve each equation.

10. Solve: $\log_8(m) = 3$
Give an exact answer. Simplify as much as possible.
m =_____

11. Solve: $\log_4(s) = 8$
s =_____

12. If $\log_2(6x + 3) = 5$, then x = _____ .

13. Solve each equation for x. If needed, first convert from exponential to logarithmic form. *Round answers to 3 decimal places. If there is no solution enter* DNE.
(a) $10^x = 9438$
x = _____
(b) $10^x = 0.002$
x = _____
(c) $10^x = -5$
x = _____
(d) $10^x = 0$
x = _____

14. Solve for x. Round answers to four decimal places.
$\ln(x) = 3$

_____

15. Solve by taking the logarithm of each side. *Round answer to 3 decimal places.*
$5^w = 1378$

$w =$ _____

16. Solve by taking the logarithm of each side. *Round answer to 3 decimal places.*

$400(1.18)^n = 27,960$

$n =$ _____

17. Application. The Richter Scale reading, R, of an earthquake is based on a logarithmic equation. Suppose:

$$R = \log\left(\frac{A}{A_0}\right)$$

where

A - is the measure of the amplitude of the earthquake wave, and

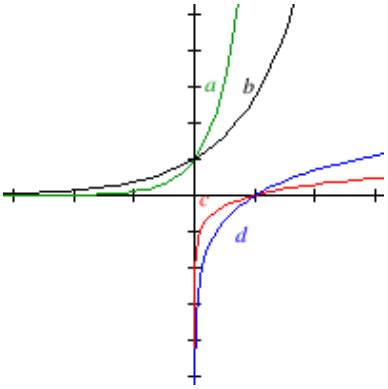
A_0 - is the amplitude of the smallest detectable wave (or standard wave).

Suppose an earthquake is measured with a wave amplitude $A = 0.3444$ while the smallest detectable wave A_0 is measured at 0.0001 cm. What is the magnitude of this earthquake using the Richter scale, to the nearest tenth?

The earthquake registered _____ on the Richter scale.

Graph Logarithms. Graph Logarithms

18.

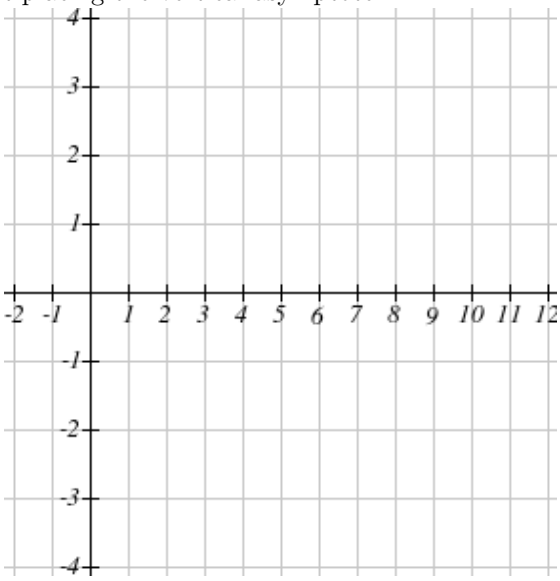


Match each equation with a graph above

1. $\ln(x)$
2. $\log(x)$
3. 10^x
4. e^x

1. green
2. black
3. red
4. blue

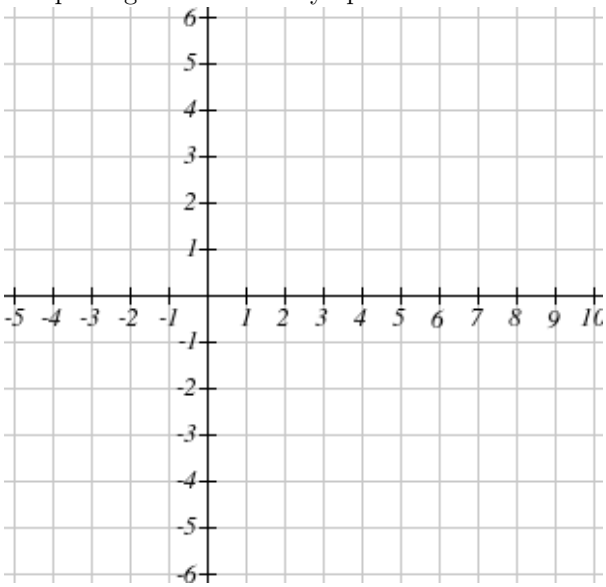
19. Graph $y = \log_8(x)$ by first placing the vertical asymptote.



20. Fill in the table of values for the function $f(x) = \log_2(x + 4)$. Enter *DNE* if the answer does not exist.

x	$f(x)$
-6	____
-3	____
-2	____

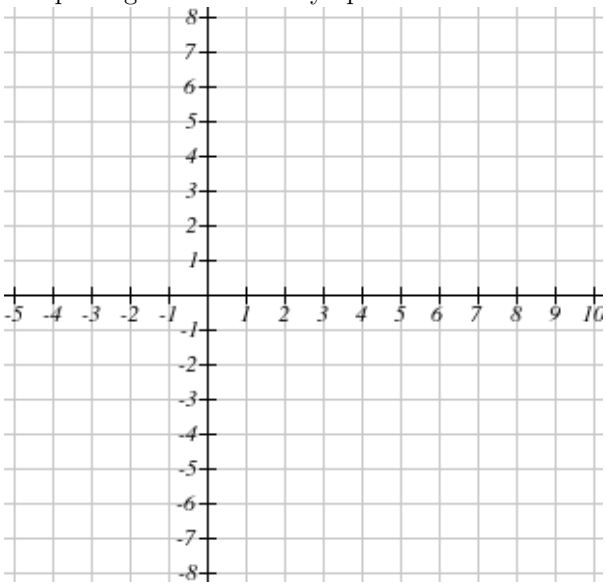
Graph the function by first placing the vertical asymptote:



21. Fill in the table of values for the function $f(x) = \log_3(x) - 1$. Enter *DNE* if the answer does not exist.

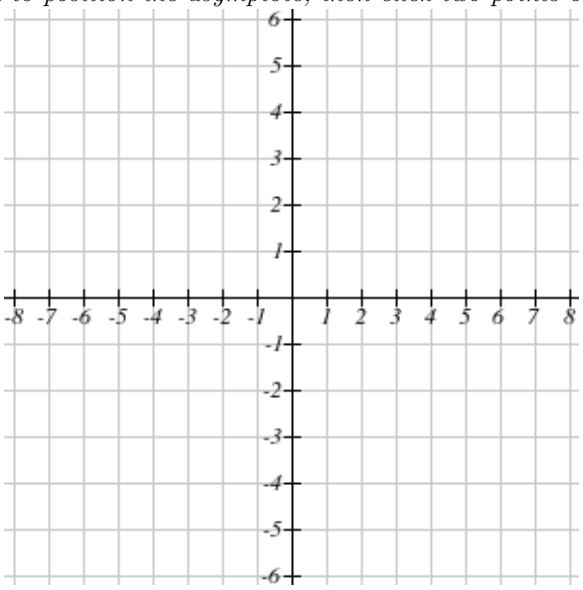
x	$f(x)$
0	____
1	____
3	____

Graph the function by first placing the vertical asymptote:



22. Graph the function $y = \log_2(x)$

Tool help: First click to position the asymptote, then click two points on the graph.



6.4 Solving Equations Using Logarithm

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)
- Read and interpret models (critical thinking)

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

This section presents methods for solving equations that have an exponential by using the relationship with the logarithm, and also how to solve some equations with logarithms.

6.4.1 Solving Equations with Exponentials

The first two examples demonstrate how we can use the inverse relationship between exponentials and logarithms to solve an equation with an exponential.

Example 6.4.1 Solve $6.0 = 10^{\frac{3}{2}(x+10.7)}$.

Order of operations dictates the following

$$6.0 = 10^{\frac{3}{2}(x+10.7)}.$$

Use a log to undo an exponential.

$$\begin{aligned}
\log(6.0) &= \log\left(10^{\frac{3}{2}(x+10.7)}\right). \\
\log(6.0) &= \frac{3}{2}(x+10.7). \\
\frac{2}{3}\log(6.0) &= x+10.7. \\
0.51876750 &\approx x+10.7. \\
0.51876750-10.7 &= x. \\
-10.18123250 &\approx x. \\
-10.181 &\approx x.
\end{aligned}$$

Rounding here was arbitrarily chosen to be 3 decimal places, because there is no context. ▲

Example 6.4.2 Solve $13 = 2 + 3e^{x-1}$.

Order of operations dictates the following

$$\begin{aligned}
13 &= 2+3e^{x-1}. \\
11 &= 3e^{x-1}. \\
\frac{11}{3} &= e^{x-1}
\end{aligned}$$

Use a ln to undo an exponential.

$$\begin{aligned}
\ln\left(\frac{11}{3}\right) &= \ln\left(e^{x-1}\right). \\
\ln\left(\frac{11}{3}\right) &= x-1. \\
1.29928298 &= x-1. \\
1.29928298+1 &= x. \\
2.29928298 &\approx x. \\
2.299 &\approx x.
\end{aligned}$$

Rounding here was arbitrarily chosen to be 3 decimal places, because there is no context. ▲

Checkpoint 6.4.3 Solve the equation: $3\left(10^{3x}\right) - 16 = 0$.

$x =$_____

Round to 3 decimal places.

The next two examples show us how to solve equations involving exponentials other than the common (base 10) or natural (base e).

Example 6.4.4 Solve $5.0 = 3.0\left(2^{\frac{t}{70}}\right)$.

Because the exponential is base 2, we will use a log base 2.

$$\begin{aligned} 5.0 &= 3.0\left(2^{\frac{t}{70}}\right). \\ 5.0/3.0 &= 2^{\frac{t}{70}}. \\ \log_2(5.0/3.0) &= \log_2\left(2^{\frac{t}{70}}\right). \\ \log_2(5.0/3.0) &= \frac{t}{70}. \\ 70\log_2(5.0/3.0) &= t. \end{aligned}$$



Standalone
In Context

▲

Your calculator likely does not have a button for calculating $\log_2(x)$. We can use a property of all logarithms to solve this equation with the natural log. In general

$$\log_b(a^p) = p\log_b(a).$$

Example 6.4.5 Solve $5.0 = 3.0\left(2^{\frac{t}{70}}\right)$. Round to units.

$$\begin{aligned} 5.0 &= 3.0\left(2^{\frac{t}{70}}\right). \\ 5.0/3.0 &= 2^{\frac{t}{70}}. \\ \ln(5.0/3.0) &= \ln\left(2^{\frac{t}{70}}\right). \\ \ln(5.0/3.0) &= \frac{t}{70}\cdot\ln(2). \\ \frac{70\ln(5.0/3.0)}{\ln(2)} &= t. \\ 51.58759159 &\approx t \\ 52 &\approx t \end{aligned}$$

Your calculator does have a button for $\ln(x)$. Note this implies that $\log_2(5.0/3.0) = \frac{\ln(5.0/3.0)}{\ln(2)}$. This relationship works for logs of any base. ▲

Checkpoint 6.4.6 radium-213 has a half-life of 2.74 seconds, and its decay is modeled by $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{2.74}}$. If the initial amount is $A_0 = 8.6$, how long will it take until only one third, approximately $A \approx 2.9$ is left?

_____

6.4.2 Solving Equations with Logarithms

As with equations involving exponentials, we can solve equations involving logarithms using the inverse relationship between exponentials and logarithms. The first two examples demonstrate using the definition of logarithm.

Example 6.4.7 Solve

$$\log_3(x) = 2.$$

We can re-write this as

$$3^2 = x$$

which tells us that $x = 9$. ▲

Example 6.4.8 Solve $14 = 12 + \log(5x)$.

First we combine the terms outside the logarithm, the we re-write that as an exponential.

$$\begin{aligned} 14 &= 12 + \log(5x) \\ 2 &= \log(5x) \end{aligned}$$

Re-write the logarithm as an exponential.

$$\begin{aligned} 10^2 &= 5x \\ 100 &= 5x \\ 20 &= x \end{aligned}$$

▲

Example 6.4.9 Solve $7 = \ln(3x + 2)$. Round to units.

Solution. We re-write this as an exponential.

$$\begin{aligned} 7 &= \ln(3x + 2) \\ e^7 &= 3x + 2 \\ e^7 - 2 &= 3x \\ \frac{1}{3}(e^7 - 2) &= x \\ 365 &\approx x \end{aligned}$$

▲

Checkpoint 6.4.10 Solve $3.5 = \left(\frac{2}{3}\right)\log(M_0) - 10.7$.

$M_0 =$ _____

6.4.3 Applications with Exponentials and Logarithms

Example 6.4.11 In Example 6.1.13 we produced the model $P = 3.0 \cdot 2^{t/7\bar{0}}$. Here we will redo this problem by converting the model to the form $P = P_0e^{rt}$. Specifically, we will find r such that $(e^r)^t = (2^{t/7\bar{0}})^t$.

The bacteria **lactobacilus acidophilus** doubles in population every 70 minutes. If the initial amount was 3 grams, what how much would there be after 24 hours?

To use this model we must first calculate the constant r. We know the amount doubles every 70 minutes, so we can start with the following.

$$\begin{aligned} 6.0 &= 3.0e^{70\bar{r}}. \\ \frac{6.0}{3.0} &= e^{70\bar{r}}. \\ 2.0 &= e^{70\bar{r}}. \\ \ln(2.0) &= \ln\left(e^{70\bar{r}}\right). \\ \ln(2.0) &= 70\bar{r}. \\ \frac{\ln(2.0)}{70\bar{}} &= r. \\ r &\approx 0.009902. \\ &\approx 0.0099. \end{aligned}$$

Using this value we can now calculate the value after 24 hours. Note 24 hours is $24 \cdot 60 = 1440$ minutes.

$$\begin{aligned} P &= 3.0e^{(1440\bar{})(0.0099)} \\ &\approx 4.674533843 \times 10^6 \\ &\approx 4.7 \times 10^6 \end{aligned}$$

This is $4.7 \times 10^3 \approx 47,000$ kilograms. Naturally, this is an unreasonable prediction. This tells us there must be other factors in bacteria growth. ▲

Example 6.4.12 Plutonium-241 has a half-life of 14.4 years. This means if you start with 10.0 g of Pu-241 in 14.4 years there will be only 5.0 g of Pu-241. Generally, this can also be modeled by

$$P = P_0 e^{rt}.$$

P_0 is the initial amount of material. P is the amount left after t units of time. r is a constant that is derives from the speed how fast the material decays.

(a) Write the model for Plutonium-241.

Solution. We must first calculate r. We can use Example 6.4.1 as an example.

$$\begin{aligned} 5.0 &= 10.0e^{r(14.4)} \\ \frac{1}{2.0} &= e^{r(14.4)} \\ \ln\left(\frac{1}{2.0}\right) &= \ln\left(e^{r(14.4)}\right) \\ \ln\left(\frac{1}{2.0}\right) &= r(14.4). \\ \frac{1}{14.4}\ln\left(\frac{1}{2.0}\right) &= r \\ -0.04813522 &\approx r. \\ -0.048 &\approx r. \end{aligned}$$

Thus the model is

$$P = P_0 e^{-0.048t}.$$

(b) If a lab has 12.0g of Pu-241, how much will be left in 6 years?

Solution. We use the model from the previous step.

$$P = 12.0e^{-0.048(6)}$$

$$\approx 8.997139$$
$$\approx 9.0.$$

▲

Note, that the example in Example 6.1.13 and Example 6.4.11 imply that $3.0 \cdot 2^{t/70} = 3.0e^{(0.0099)t}$. In particular we can convert the power of 2 to a power of e. More generally, we can write $2^x = e^{rx}$ or $3^x = e^{rx}$ or similar for some value of r. The next example shows how we can perform this conversion.

Example 6.4.13 Write 2^x as e^{rx}. Numbers are exact.

$$\begin{aligned} 2^x &= e^{rx}. \\ \ln(2^x) &= \ln(e^{rx}). \\ x\ln(2) &= (rx)\ln(e). \\ x\ln(2) &= rx. \\ \frac{x\ln(2)}{x} &= \frac{rx}{x}. \\ \ln(2) &= r \end{aligned}$$

Thus $2^x = e^{x\ln(2)}$. ▲

Definition 6.4.14 pH (percent hydrogen). Acidity is measured in pH (percent hydrogen). The calculation is

$$\text{pH} = -\log(H_3O^+)$$

where H_3O^+ is the concentration of hydronium ions per mole. This is obtained experimentally. ◆

Example 6.4.15

(a) A solution of hydrochloric acid has a concentration of 0.0025. Calculate its pH.

$$\begin{aligned} \text{pH} &= -\log(0.0025) \\ &\approx 2.6. \end{aligned}$$

(b) Sweat has a pH between 4.5 and 7. Suppose sweat is measured to have a pH of 5.3. Determine the concentration of ions.

We setup the pH calculation and solve using Example 6.4.9.

$$\begin{aligned} 5.3 &= -\log(c) \\ -5.3 &= \log(c) \\ 10^{-5.3} &= 10^{\log(c)} \\ 10^{-5.3} &= c \\ 5.0 \times 10^{-6} &\approx c \end{aligned}$$

▲

What is the purpose behind defining pH using a log? What does it do, that simply giving the concentration of H_3O^+ does not? The next example illustrates what the use of a log adds.

Example 6.4.16

(a) Calculate the concentration of H_3O^+ for pH levels of 5, 6, 7 (all within the range of human sweat).

Solution.

$$\begin{aligned} 5 &= -\log(c_5). \\ 10^{-5} &= c_5. \\ 6 &= -\log(c_6). \\ 10^{-6} &= c_6. \\ 7 &= -\log(c_7). \\ 10^{-7} &= c_7. \end{aligned}$$

(b) What is the ratio of the concentration from pH of 5 to 6? of 6 to 7? of 5 to 7? This means calculate the ratios c_5/c_6, c_6/c_7, and c_5/c_7.

Solution.

$$\begin{aligned} \frac{c_5}{c_6} &= \frac{10^{-5}}{10^{-6}} = 10. \\ \frac{c_6}{c_7} &= \frac{10^{-6}}{10^{-7}} = 10. \\ \frac{c_5}{c_7} &= \frac{10^{-5}}{10^{-7}} = 100. \end{aligned}$$

▲

We see that a change of one pH means the substance is 10 times as strong. A change of 2 pH means the substance is $10^2 = 100$ times as strong. The log scale gives us growth as a ratio.

Definition 6.4.17 Moment Magnitude. Larger earthquakes today are measured and reported using the **moment magnitude** scale. This is calculated via

$$M_w = \frac{2}{3}\log(M_0) - 10.7$$

where M_w is the moment magnitude, and M_0 is the seismic moment in Newtons per meter (a measure of energy). ◆

Example 6.4.18

(a) Based on seismic readings $M_0 = 7.2 \times 10^{22}$. What was the moment magnitude? These are rounded to one decimal place.

Solution. Using the formula we obtain

$$\begin{aligned} M_w &= \frac{2}{3}\log(7.2 \times 10^{22}) - 10.7 \\ &\approx 4.53822167 \\ &\approx 4.5 \end{aligned}$$

(b) What was the seismic moment for an earthquake with magnitude 7.1?

Solution. We setup the calculation and solve using Example 6.4.9.

$$\begin{aligned} 7.1 &= \frac{2}{3}\log(M_0) - 10.7 \\ 17.8 &= \frac{2}{3}\log(M_0) \\ 26.7 &= \log(M_0) \\ 10^{26.7} &= M_0 \end{aligned}$$

$$5.011872336 \times 10^{26} \approx M_0$$
$$5.01 \times 10^{26} \approx M_0$$

▲

Just as with pH, the moment magnitude seismic scale enables us to compare how much stronger one earthquake is than another. The 2/3 means that a change of 1 is not 10 times as strong but a different ratio. Can you figure out what that ratio is?

6.4.4 Exercises

Solve Exponential and Logarithmic Equations. Solve each equation using the definitions or devices as directed.

1. Solve for x: $4^x = 53$
$x =$ _____

2. Find the solution of the exponential equation
$100(1.02)^{2t} = 50,000$
in terms of logarithms, or correct to four decimal places.
$t =$ _____

3. Find the solution of the exponential equation
$3e^x = 10$
in terms of logarithms, or correct to four decimal places.
$x =$ _____

4. Find the solution of the exponential equation
$e^{1-4x} = 7$
in terms of logarithms, or correct to four decimal places.
$x =$ _____

5. Solve for x:
$2^{7x-2} = 4^{8x-6}$
$x =$ _____ .

6. Solve.
$\log_6(v) = 8$
$v =$_____

7. Solve *exactly.*
$\ln(-8x) = 75$
$x =$_____

8. Solve *exactly.*
$18\log_{17}(x) = 11$
$x =$_____

9. Solve for x
$\log_3(2x - 6) = 3$
_____

Applications. Use logarithms and exponentials to answer questions about these applications.

10. The pH scale for acidity is defined by $\text{pH} = -\log_{10}[\text{H}^+]$ where $[\text{H}^+]$is the concentration of hydrogen ions measured in moles per liter (M).
A solution has a pH of 4.55.
Calculate the concentration of hydrogen ions in moles per liter (M).
The concentration of hydrogen ions is _____ moles per liter.

11. The number of bacteria in a culture is given by the function
$n(t) = 970e^{0.5t}$
where t is measured in hours.
(a) What is the relative rate of growth of this bacterium population?
Your answer is _____ percent
(b) What is the initial population of the culture (at t=0)?
Your answer is _____
(c) How many bacteria will the culture contain at time t=5?
Your answer is _____

12. A population of bacteria is growing according to the equation $P(t) = 550e^{0.14t}$. Estimate when the population will exceed 1274.
t= _____
Give your answer accurate to one decimal place.

13. An unknown radioactive element decays into non-radioactive substances. In 480 days the radioactivity of a sample decreases by 32 percent.
(a) What is the half-life of the element?
half-life: _____ (days)
(b) How long will it take for a sample of 100 mg to decay to 65 mg?
time needed: _____ (days)

14. A cell of some bacteria divides into two cells every 40 minutes.The initial population is 3 bacteria.
(a) Find the size of the population after t hours
$y(t)$ = _____
(function of t)
(b) Find the size of the population after 3 hours.
$y(3)$ = _____
(c) When will the population reach 9?
T = _____

15. Diseases tend to spread according to the exponential growth model. In the early days of AIDS, the growth factor (i.e. common ratio; growth multiplier) was around 1.9. In 1983, about 1800 people in the U.S. died of AIDS. If the trend had continued unchecked, how many people would have died from AIDS in 2005?
_____ people
(Note: once diseases become widespread, they start to behave more like logistic growth, but don't worry about that for the purpose of this exercise)

16. A native wolf species has been reintroduced into a national forest. Originally 250 wolves were transplanted, and after 3 years the population had grown to 785 wolves. If the population grows exponentially according to the formula $P_t = P_0(1 + r)^t$.
(a) Find the growth rate. Round your answer to the nearest tenth of a percent.
r = _____%
(b) If this trend continues, how many wolves will there be in ten years?
_____ wolves
(c) If this trend continues, how long will it take for the population to grow to 1000 wolves? Round your answer to the nearest tenth of a year.
_____ years

17. A wooden artifact from an ancient tomb contains 40 percent of the carbon-14 that is present in living trees. How long ago, to the nearest year, was the artifact made? (The half-life of carbon-14 is 5730 years.)
_____ years.

18. The half-life of strontium-90 is 28 years. How long will it take a 80 mg sample to decay to a mass of 20 mg?
Your answer is _____ years.

19. The half-life of Palladium-100 is 4 days. After 16 days a sample of Palladium-100 has been reduced to a mass of 5 mg.

What was the initial mass (in mg) of the sample? _____

What is the mass 8 weeks after the start? _____

20. The half-life of caffeine in the human body is about 6.3 hours. A cup of coffee has about 105 mg of caffeine.

1. Write an equation for the amount of caffeine in a person's body after drinking a cup of coffee? Let C be the milligrams of caffeine in the body after t hours. _____
2. How much caffeine will remain after 10 hours? ___ mg. State your answer to the nearest hundredth of a mg.
3. How long until there are only 20 mg remaining? ___ hours. State your answer to the nearest hundredth of an hour.

21. Many substances are metabolized by our body so that the amount of the substance in our system decreases exponentially. Sometimes studies state this in terms of "half-life", and sometimes as an hourly rate of decrease.

$A \cdot \left(\frac{1}{2}\right)^{\frac{t}{HL}} = A \cdot (1-r)^t$

Where A is the initial amount, HL is the half-life in hours, t is the time in hours, and r is the hourly rate of decrease.

Suppose a substance has a half-life of 6 hours.

What is the hourly decay rate, to the nearest tenth of a percent? _____ %

Hint: Consider a starting amount 1 after 1 hour.

22. Vasya and Tien are hired at different companies but both receive an initial salary of $59500. Each year Vasya receives a raise of $2113. Tien receives a raise that is 3.24 %.

After how many years is Vasya's salary at least $100,000? ____

After how many years is Tien's salary at least $100,000? ____

Who reaches $100,000 sooner?

1. Tien
2. Vasya
3. the same

23. An approximation of the speed at which a vehicle will hydroplane on water (complete loss of traction) is

$V = 10.2\sqrt{p}$

where

V is the velocity in miles per hour

p is the tire pressure in psi

You plan to travel no faster than 65.0 miles per hour. To be somewhat safe from hydroplaning, what pressure should your tires be over? ____ psi

These are science measurements. Round appropriately.

6.5 Project: Time of Death

Project 6.5.1 Estimating Time of Death. The purpose of this project is to practice reading a mathematical model, using it to calculate a result, and interpreting its features. This model involves an exponential relation.

If a person is believed to have died within a day or so of the body's discovery, it's possible to estimate

the time of death using body temperatures. Isaac Newton's idea was that since hot things cool much faster than cool things, the rate of cooling is more or less proportional to the temperature of the object, resulting in an exponential decay model.

Theorem 6.5.2 Newton's Law of Cooling. *The temperature T at a time x of a cooling object follows the function*

$$T = A + Be^{kx}.$$

A is the ambient (or room) temperature. B and k are constants that depend on the object.

Suppose a forensics technician arrived at a murder scene and recorded the temperature of the surroundings as well as the body. The technician decides it is fair to assume that the room temperature has been holding steady at about 68 °F. A thermometer was placed in the liver of the corpse and the following table of values was recorded.

Table 6.5.3 Time and Temperature

Actual Time	Minutes Elapsed (x)	Temperature, T, of the Body (°F)
2:00 pm	0	85.90
2:20 pm	20	85.17
2:40 pm	40	84.47
3:00 pm	60	83.78

The key to estimating time of death is to estimate A, B, k, and x in Theorem 6.5.2.

(a) Recall that the technician thinks the room temp was 68 °F. By substituting this number and the first recording (0, 85.9) into the Cooling Equation, calculate $A =$ _____ and $B =$ _____.

(b) Once you know A and B, substitute some *other* data point into the equation so that k is the only variable. Solve the resulting equation and round k to 6 decimal places. Show your work. Remember the exponential must be isolated before you take the natural log of both sides.

$$k \approx$$ _____

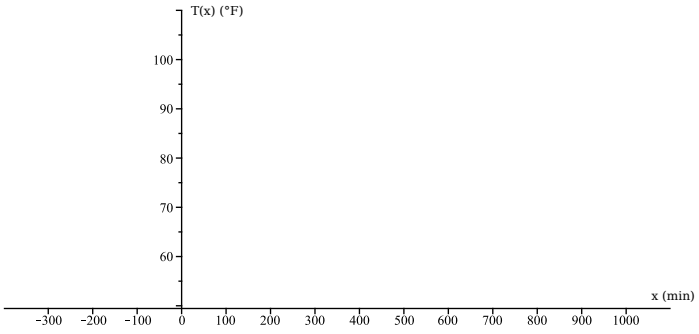
(c) Note: the number k is called the cooling (or warming) constant. If an object cools, k should be negative. Mathematically, looking at the equation, why should k be negative?

(d) Using the numbers you found for A, B, and k, write the equation for the temperature at any time x.

$T(x) =$ _____

(e) Draw a graph of the temperature function $T(x)$. Completing the table of values may help you graph.

x	y
-300	
-100	
100	
300	
500	
700	
900	



(f) The graph of $T(x)$ has a horizontal asymptote. What is the height of this asymptote and what does it tell you about the way corpses cool?

(g) Notice that this equation deviates from reality if the x-value goes too far negative. Generally speaking, (no numbers required), at what point does the model no longer work? What in reality gives us an indication that we've taken it too far?

(h) Assuming that the temperature of the person at the time of death (TOD) was 98.6 °F, set up a TOD equation using the values of A, B, and k you've calculated. Then, solve the equation using the same logarithm method you used to solve for k.

Write your answer as a time, not just as x minutes. Recall that when $x = 0$, the time is 2:00 PM.

(i) When a forensic expert determines time of death, they often have additional information besides body temperature. Suppose a coroner finds that the person who was murdered had an infection that probably raised the core body temperature to around 102 °F. Using the same cooling constant, ambient room temperature, and temperature data as in Task 6.5.1.a and Task 6.5.1.b, make a new estimate for the time of death.

Again, write your answer as a time, not just as x minutes.

Chapter 7

Trigonometry

7.1 Trigonometric Ratios

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
- Analyze right triangles (skill)
- Identify properties of sine and cosine functions (skill)

Section 4.1 presented areas of triangles and a relationship between the three sides of a right triangle. This section and the following ones present relationships between angles of triangles and the lengths of their sides.

7.1.1 Angle Relationships in Triangles

The Pythagorean Theorem states a relationship between the side lengths of all right triangles. There is also a relationship between the three angles of all (not just right) triangles.

Theorem 7.1.1 Triangle Angle Sum. *The sum of the angles of any triangle is* $180°$.

Example 7.1.2 If two angles of a triangle are $40°$ and $70°$ what is the other angle?

The third angle must satisfy

$$\begin{aligned} 40° + 70° + \theta &= 180°. \\ \theta &= 70°. \end{aligned}$$

▲

Example 7.1.3 If one angle of a right triangle is $55°$ what is the other angle?

Solution. We know the measure of one angle is $90°$ (right angle) and the measure of another angle is $55°$. Thus the third angle must satisfy

$$\begin{aligned} 90° + 55° + \theta &= 180°. \\ \theta &= 35°. \end{aligned}$$

▲

Checkpoint 7.1.4 If two angles of a triangle have measures 89° and 55°, what is the measure of the third angle? ____°

For a right triangle each of the other two angles have measure less than a right angle. This is a result of the Triangle Angle Sum theorem. $180° - 90° = 90°$ so the remaining two angles have a sum that adds to 90°. This implies both non-right angles are smaller than a right angle.

7.1.2 Defining Trig Functions

This section presents the trigonometric functions and demonstrates that they make sense. First, we need terminology with which to describe right triangles.

For right triangles we have names for the sides. Consider the labels in Figure 7.1.5. These names are relative to the particular non-right angles with which we are working. In this case it is the one labeled α. The **adjacent** is the side that connects the angle α and the right angle. The **opposite** is the side touching the right angle but not touching the angle (α). Both the adjacent and opposite sides are known as **legs** of the right triangle. The **hypotenuse** is opposite the right angle (the one side not touching it).

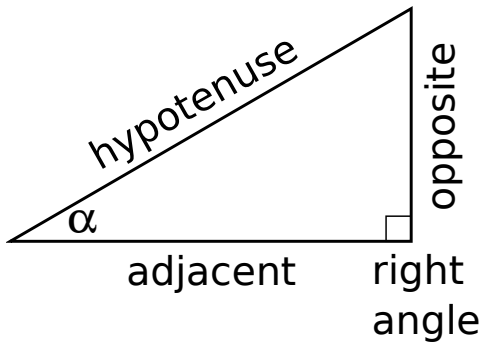


Figure 7.1.5 Right Triangle Terminology

Example 7.1.6 Consider the triangle in Figure 7.1.8. With respect to the angle α the adjacent side is the one with length 5, and the opposite side is the one with length 3.

With respect to the angle θ the adjacent side is the one with length 3, and the opposite side is the one with length 5. ▲

The trigonometric functions are defined below as ratios of side lengths. We will use only the first three in this text.

Table 7.1.7 Trig Functions as Ratios

sine	$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}}$
cosine	$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$
tangent	$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}}$
secant	$\sec(\alpha) = \frac{\text{hypotenuse}}{\text{adjacent}}$
cosecant	$\csc(\alpha) = \frac{\text{hypotenuse}}{\text{opposite}}$
cotangent	$\cot(\alpha) = \frac{\text{adjacent}}{\text{opposite}}$

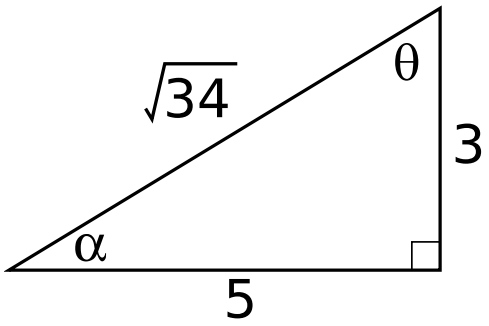


Figure 7.1.8 Right Triangle with Side Lengths

Example 7.1.9 Given the side lengths in Figure 7.1.8 what are each of the following trig ratios?

(a) $\sin(\alpha) =$

From the perspective of α, the opposite side has length 3. The hypotenuse has length $\sqrt{34}$. Thus

$$\sin(\alpha) = \frac{3}{\sqrt{34}}$$

(b) $\cos(\alpha) =$

From the perspective of α, the adjacent side has length 5. The hypotenuse has length $\sqrt{34}$. Thus

$$\cos(\alpha) = \frac{5}{\sqrt{34}}$$

(c) $\sec(\alpha) =$

$\sec(\alpha)$ flips the ratio of $\cos(\alpha) = \frac{5}{\sqrt{34}}$ (from the previous problem). Thus

$$\sec(\alpha) = \frac{\sqrt{34}}{3}$$

(d) $\sin(\theta) =$

From the perspective of θ, the opposite side has length 5. The hypotenuse has length $\sqrt{34}$. Thus

$$\sin(\theta) = \frac{5}{\sqrt{34}}$$

▲

7.1.3 Solving Triangles

Now that we have defined trigonometric functions, we can use them to analyze triangles. The goal is to calculate all of the side lengths and/or angles given only some of them.

Example 7.1.10 For a right triangle with angle $\alpha = 50°$ and corresponding opposite side of length 7, what are the other side lengths and angles?

First, we know that two of the angles are 90° and 50°, so the third angle has measure $180° - 90° - 50° = 40°$.

To calculate the length of the hypotenuse recall that $\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}}$. We know the angle and the length of the opposite.

$$\sin(50°) \approx 0.76604444. \qquad \text{Use a device to approximate.}$$

$$\frac{7}{h} \approx 0.76604444. \qquad \text{Write the ratio.}$$

$$7 \approx h \cdot 0.76604444. \qquad \text{Clear the denominator.}$$
$$\frac{7}{0.76604444} \approx h.$$
$$h \approx 9.1378510.$$

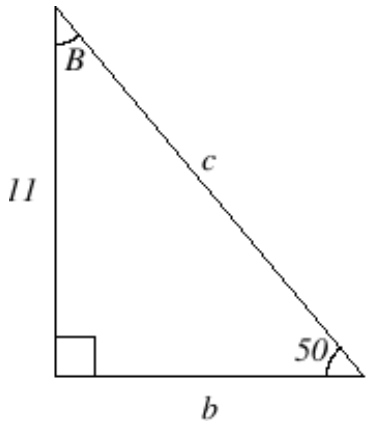
The hypotenuse has length 9.13. Now that we know two sides we can use the Pythagorean Theorem to calculate the length of the adjacent.

$$7^2 + b^2 = 9.1378510^2.$$
$$b^2 = 9.1378510^2 - 7^2.$$
$$b^2 = 34.500321.$$
$$\sqrt{b^2} = \sqrt{34.500321}.$$
$$b \approx 5.8736974.$$

The adjacent has length 5.87.

Rounding was arbitrarily chosen to be two (2) decimal places, because we have no context. ▲

Checkpoint 7.1.11



Suppose a = 11 and A = 50 degrees. Calculate:
$b \approx$ ____
$c \approx$ ____
$B \approx$ ____ degrees
Round all answers to one decimal place. Give angles in *degrees*

7.1.4 Calculating lengths using trig functions

This section demonstrates using trigonometric functions to calculate lengths in an application when we know an angle and a side length.

First, we define terminology we need to describe the applications.

Definition 7.1.12 Angle of Elevation. The **angle of elevation** of an object or observation is the angle measured from level (often the ground) up to the object (or line of sight). ◆

Definition 7.1.13 Angle of Depression. The **angle of depression** of an object or observation is the angle measured from level down to the object (or line of sight). ◆

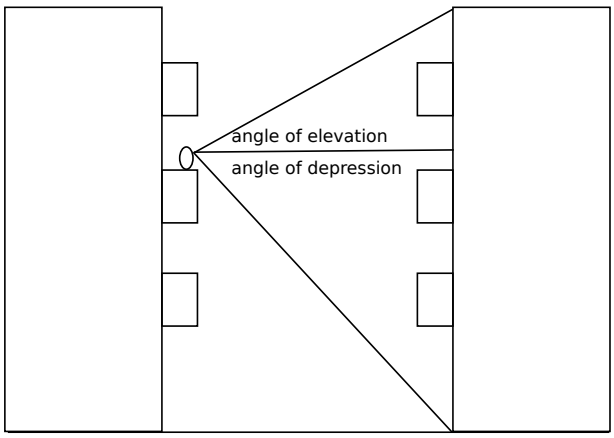
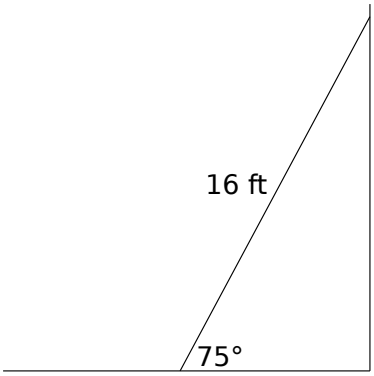


Figure 7.1.14 Illustrations of Angles of Elevation and Depression

For all of these applications our first task is to recognize a right triangle in the problem. We must also identify what the two legs and/or the hypotenuse are in the application. Then we can set up an equation using a trigonometric function, and use the equation to calculate something.

Example 7.1.15 For safety reasons the optimal angle of elevation of a ladder is 75°. If the ladder is 16 ft long, at what height will the top of the ladder be resting against a wall? We can measure a tenth of a foot but not very easily measure a hundredth of a foot, especially for placing a ladder.

First, it is often useful to sketch an image. This makes it easier to identify triangles or other shapes. Note the sketch does not need to be artistic.



We notice that the ladder forms the hypotenuse of a right triangle with the ground and the wall. Next we identify details. We know an angle (75° angle of elevation from the ground) and the length of the hypotenuse (length of the ladder). We want the length of the side opposite the angle (height along the wall). From this information (opposite, hypotenuse, angle) we can recognize the need for the sine function.

$$\begin{aligned} \sin(\theta) &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \sin(75^\circ) &= \frac{h}{16} \\ 16\sin(75^\circ) &= h \\ 15.45 &\approx B \end{aligned}$$

Thus the top of the ladder is 15.5 feet up the wall.

Why round to one decimal place? A hundredth of a foot is much less than an inch. This would make no difference in any application. Rounding to the nearest foot (units) would ignore as much as a half foot which could make a difference. ▲

Example 7.1.16 We may also wish to know how far from the wall to place the bottom of the ladder. That is calculating the length of the side adjacent to the angle of elevation, so we use the cosine function.

$$\begin{aligned} \cos(75^\circ) &= \frac{A}{16} \\ 16\cos(75^\circ) &= A \\ 4.14 &\approx A \end{aligned}$$

Thus we place the ladder a little more than 4 feet from the wall.

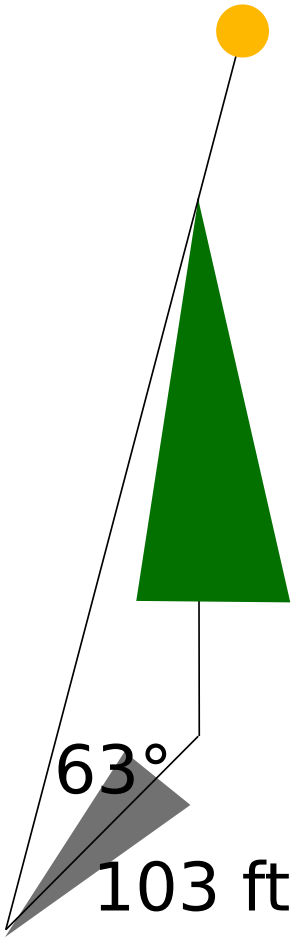
If we had already calculated the height up the wall (previous example) we could also use the Pythagorean Theorem.

$$\begin{aligned} A^2 + 15.45^2 &= 16^2 \\ A^2 &= 16^2 - 15.45^2 \\ A^2 &= 256 - 238.7025 \\ A^2 &= 17.2975 \\ A &= \sqrt{17.2975} \\ &\approx 4.1590 \\ &\approx 4.16 \end{aligned}$$

This is quite close to the previous solution, specifically the difference is too small to effect ladder placement. The difference is the result of using the 15.45 length which was rounded. ▲

Example 7.1.17 Measuring the heights of tall objects is a use of trigonometry that has been around for millenia.

We want to determine the height of a tree, but dropping a measuring tape from the top is impractical. Instead we can use its shadow, which being on the ground, is easier to access.



The sun casts a shadow of a tree on the ground. The distance from the top of the shadow to the base of the tree is measured to be 103 ft (this is the length of the shadow). From the end of the shadow the angle of elevation to the sun is measured to be 63°. How tall is the tree? Round using significant digits because this is based on measurements. We do not want to claim a precision about the height which is not valid.

The tree, shadow, and line to the sun form a right triangle with angle 63°, and an adjacent side length of 103 ft. The height of the tree is the length of the opposite leg. Because we know the adjacent and want the opposite we use the tangent function.

$$\begin{aligned} \tan(\theta) &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(63°) &= \frac{H}{103} \\ 103\tan(63°) &= H \\ 2\underline{0}2.15 &\approx H \\ 2\bar{0}0 &\approx H \end{aligned}$$

The tree is approximately 200 feet high. ▲

Just as in Significant Digits beyond Arithmetic and Significant Digits beyond Arithmetic trigonometric functions and their inverses can be calculated to preserve the same number of significant digits.

Example 7.1.18 Aircraft typically fly a 3° angle of depression to a point 1020 ft past the start of the runway. How high would the plane be when it crosses the runway threshhold? Round to the nearest foot, because aircraft cannot be controlled sufficiently precisely for greater precision to matter here.

Solution. This is a right triangle with adjacent leg length 1020 ft and angle 3°. The length of the opposite

is the height at the threshold.

$$\begin{aligned} \tan(3^\circ) &= \frac{T}{1020} \\ 1020\tan(3^\circ) &= T \\ 53.4559 &\approx T \\ 53 &\approx T \end{aligned}$$

▲

Checkpoint 7.1.19 At a particular airport the angle of depression flown by aircraft following the visual descent angle (VDA) is 2.86°. This leads to a point 1020 ft from the threshold of the runway. How high is the aircraft at the threshold? ____

Round to units. This height is called the Threshhold Crossing Height (TCH).

7.1.5 Limitations on Triangle Side Lengths

We should always check that results of calculations make sense. This section presents limitations on triangles we can use for these reality checks.

It is possible to define the trigonometric functions as ratios of sides, because there is a connection between how big an angle is, and how big the side across from it must be. The following activity illustrates how the three main trigonometric functions change as the angle increases or decreases because of this connection between angle measure and length of sides.

Activity 7.1.20 This activity has two steps. First notice the relationship between the how big an angle is and how long the side opposite that angle is. Second notice how the trig functions change as a result of this first fact. Use the activity in Figure 7.1.21

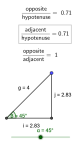
(a) Use the slider for α to increase the angle from 0° to 90°. As the angle increases what does the length of the opposite side (j) do?

(b) Use the slider for α to increase the angle from 0° to 90°. As the angle increases what does the length of the adjacent side (j) do?

(c) Note that the hypotenuse does not change in this example. Based on your result in Task 7.1.20.a, as the angle α increases what will the ratio of opposite to hypotenuse (j/g) do?

(d) Note that the hypotenuse does not change in this example. Based on your result in Task 7.1.20.b, as the angle α increases what will the ratio of adjacent to hypotenuse (i/g) do?

(e) Based on your result in Task 7.1.20.a and Task 7.1.20.b, as the angle α increases what will the ratio of opposite to adjacent (j/i) do?



Standalone
In Context
Embed

Figure 7.1.21 Sides vs Angles

While angle size and triangle side length is connected, we can always scale a triangle (e.g., make it twice

as large) without changing the angles and hence not changing the trig function values. This is why the trig functions are defined as ratios: the scale is divided out.

Another way to look at this is to recall similar triangles (Subsection 2.4.3). The ratios between corresponding sides of two, similar triangles is fixed (all three ratios are the same value). This means ratios of sides of one triangle will be the same as ratios of sides of the other triangle: one will be expressed in non-reduced form. The next examples illustrate this idea.

Example 7.1.22 Consider a right triangle with side lengths 8, 15, and 17. If A is the angle opposite from the side of length 8, then $\sin(A) = \frac{8}{17}$.

Next consider a right triangle with side lengths 16, 30, and 34. If A' is the angle across from 16, then $\sin(A') = \frac{16}{34} = \frac{8}{17}$. This is the same ratio as the previous triangle although the triangle is larger (double in each side length). ▲

Example 7.1.23 Consider the right triangle with side lengths 8, 15, and 17. If we scale this triangle until the side of length 8 is now length 40, what are the other side lengths?

If A is the angle opposite from the side of length 8 in the original triangle, then $\sin(A) = \frac{8}{17}$. Scaling the triangle does not change the angles, so the new triangle has an angle with the same angle measure as A, call it A'. Putting these together gives us the following.

$$\begin{aligned} \sin(A) &= \sin(A'). \\ \frac{8}{17} &= \frac{40}{h}. \\ h &= \frac{40 \cdot 17}{8} \\ &= 85 \\ &= 17 \cdot 5. \end{aligned}$$

The last line is included to show how the hypotenuse is scaled the same way the opposite leg is. We could use tangent to show the other leg is also scaled by 5, that is the adjacent leg will be length $15 \cdot 5 = 75$. ▲

The Pythagorean Theorem tells us that if we know two sides of a right triangle, the length of the third side is already determined. This means there are restrictions on the side lengths from which a right triangle can be assembled.

Activity 7.1.24 Considering the Shortest Hypotenuse. By calculating the sides of a triangle, we will recognize a limitation on how small a hypotenuse can be relative to either leg. Treat all numbers as exact. Do not round.

(a) For a right triangle with a leg of length 4 and hypotenuse of length 5, what is the length of the other leg?

(b) For a right triangle with a leg of length 4 and hypotenuse of length 4.5, what is the length of the other leg?

(c) For a right triangle with a leg of length 4 and hypotenuse of length 4.1, what is the length of the other leg?

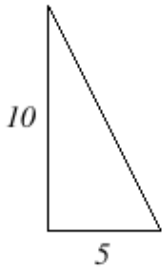
(d) For a right triangle with a leg of length 4 and hypotenuse of length 3.5, what is the length of the other leg?

(e) For a right triangle with the longest leg of length 4, how small can the hypotenuse become?

7.1.6 Exercises

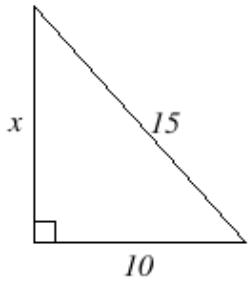
Angle Sum and Pythagorean Theorem. Use the Pythagorean Theorem and angle sum fact to calculate requested information.

1. Find the length of the hypotenuse of the triangle pictured below. Give your answer accurate to at least 2 decimal places.



hypotenuse = _____

2. Find the length of the leg x, as an exact value or to at least 2 decimal places.



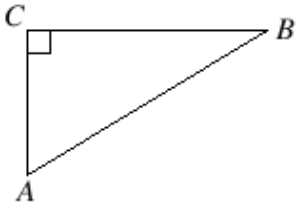
$x =$ _____

3. The measures of two angles of a triangle are 66 and 73. Find the measure of the third angle.
$\angle C =$ ____°

4. The measures of two angles of a triangle are 66 and 73. Find the measure of the third angle.
$\angle C =$ ____°

Trigonmetric Function Definition. Use the ratio definitions of trigonometric functions to answer these questions.

5.



Match each side as hypotenuse, opposite, or adjacent of angle B.

1. Opposite

2. Hypotenuse
3. Adjacent

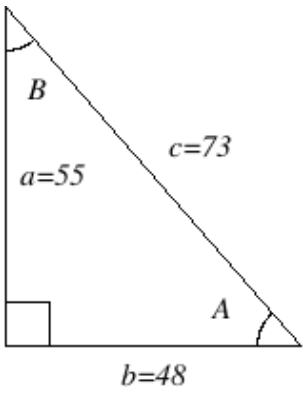
1. AC
2. BC
3. AB

6. Match each trig function with its ratio.

1. Cosine
2. Sine
3. Tangent

1. $\frac{hyp}{opp}$
2. $\frac{adj}{opp}$
3. $\frac{opp}{hyp}$
4. $\frac{hyp}{adj}$
5. $\frac{opp}{adj}$
6. $\frac{adj}{hyp}$

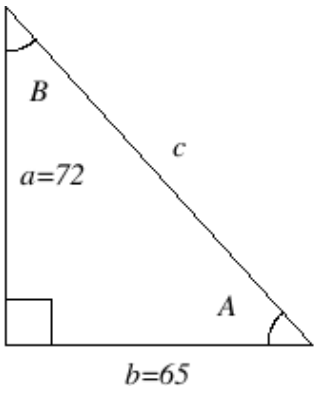
7.



Suppose the legs have lengths $a = 55$ and $b = 48$ and the hypotenuse has length $c = 73$. Answers must be exact results (fractions).

$\sin(A) =$ _____
$\cos(A) =$ _____
$\tan(A) =$ _____
$\sec(A) =$ _____
$\csc(A) =$ _____
$\cot(A) =$ _____

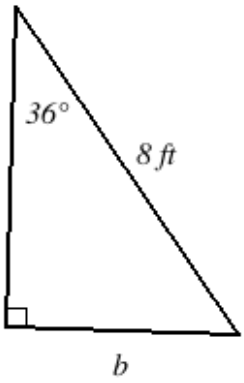
8.



Suppose the legs have lengths $a = 72$ and $b = 65$.
Answers must be exact results (fractions).
$\sin(A) =$ _____
$\cos(A) =$ _____
$\tan(A) =$ _____
$\sec(A) =$ _____
$\csc(A) =$ _____
$\cot(A) =$ _____

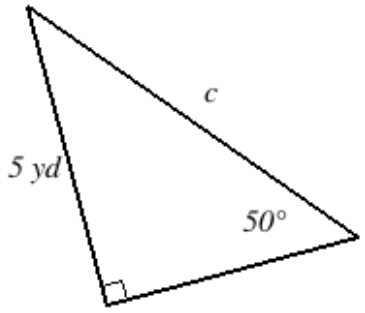
Solving Right Triangles. Calculate side lengths and angles using trigonometric functions.

9. For the right triangle below, find the length of b.



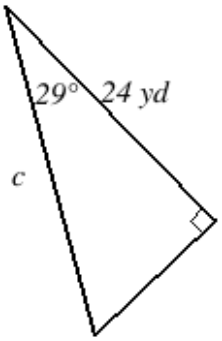
Enter the value for b (accurate to at least two decimal places) and include the units of measure.
$b =$ _____

10. For the right triangle below, find the length of c.



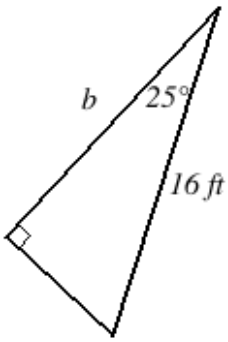
Enter the value for c (accurate to at least two decimal places) and include the units of measure.
$c =$ _____

11. For the right triangle below, find the length of c.



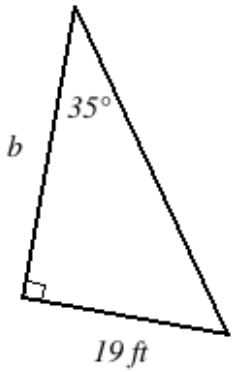
Enter the value for c (accurate to at least two decimal places) and include the units of measure.
$c =$ _____

12. For the right triangle below, find the length of b.



Enter the value for b (accurate to at least two decimal places) and include the units of measure.
$b =$ _____

13. For the right triangle below, find the length of b.

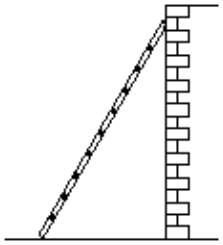


Enter the value for b (accurate to at least two decimal places) and include the units of measure.
$b =$ _____

14. What is the height of a right triangle with an angle that measures 21 degrees and an adjacent side of length 10. Enter your answer accurate to 2 decimal places.
Height $\approx$ _____

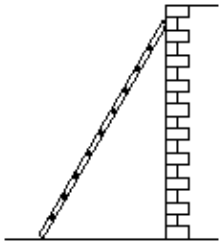
Applications. Identify a triangle in each problem then select an appropriate trigonometric function to answer these application questions.

15. The proper angle for a ladder is about 75° from the ground. Suppose you have a 17 foot ladder. How far from the house should you place the base of the ladder? Round to the nearest 10th of a foot



_____ feet

16. The proper angle for a ladder is about 75° from the ground. Suppose you have a 19 foot ladder. How high can it reach?



_____ feet
Make your answer accurate to at least 2 decimal places.

17. A smokestack is 150 feet high. A guy wire must be fastened to the stack 10 feet from the top. The guy wire makes an angle of 40° with the ground. Find the length of the guy wire rounded to the nearest foot.

_____ feet

18. The angle of elevation to the top of a Building in New York is found to be 8 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building. Round to the nearest whole number.

Your answer is _____ feet.

19. A radio tower is located 425 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 35° and that the angle of depression to the bottom of the tower is 22°. How tall is the tower?

_____ feet

Give your answer rounded to the nearest foot.

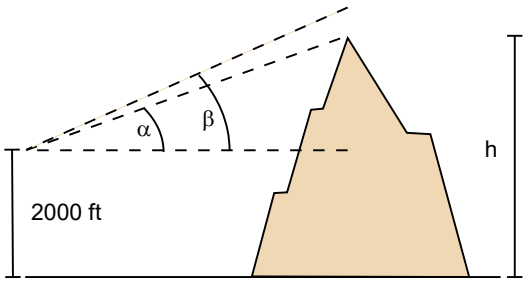
20. From the top of a 224-ft lighthouse, the angle of depression to a ship in the ocean is 33°. How far is the ship from the base of the lighthouse? Give your answer to the nearest foot.

The boat is _____ feet from the base of the light house.

21. An 28 ft extension ladder is 196 in long before extending. If each rung is 14 in above the previous, and the ladder is leaned at an angle of 75°, how much higher is the top of the ladder if it is extended 1 rungs? _____

Round down to the nearest inch.

22.



A plane is flying 2000 feet above sea level toward a mountain as shown. The pilot observes the top of the mountain to be $\alpha = 17°$ above the horizontal, then immediately flies the plane at an angle of $\beta = 20°$ above horizontal. The airspeed of the plane is 100 mph. After 5 minutes, the plane is directly above the top of the mountain. How high is the plane above the top of the mountain (when it passes over)? What is the height of the mountain? Round to the nearest 10 feet.

The height of plane above mountain is _____ feet

The height of the mountain is _____ feet

23. From the observation deck of the lighthouse at Sasquatch Point 51 feet above the surface of Lake Ippizuti, a lifeguard spots a boat out on the lake sailing directly toward the light house. The first sighting had a angle of depression of 7.5º and the second sighting had an angle of depression of 25.6º. How far had the boat traveled between the sightings?

_____ ft

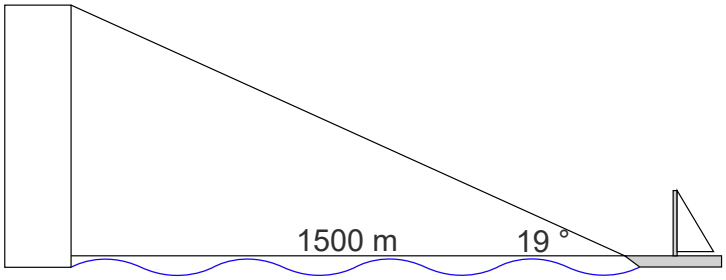
24. Below is a picture of a lean-to greenhouse. The angle of elevation of the roof is 23°. The width is 4 feet and the height from the ground to the low part of the roof is 6 feet. What is the height from

the ground to the top of the roof?



23°The height from the ground to the top of the roof is _____ feet.
Round to appropriate significant figures.

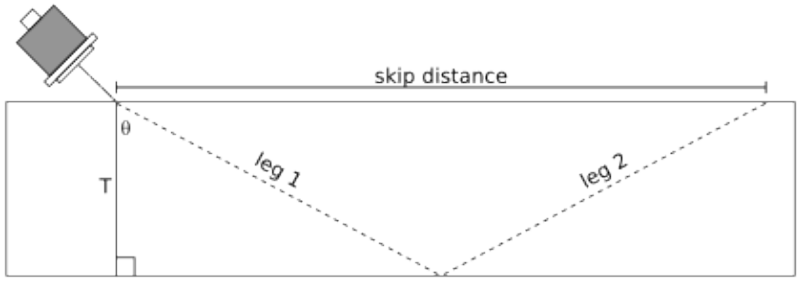
25. A boat is 1,500 meters from a cliff. If the angle of inclination from the boat to the top of the cliff is 19°, how tall is the cliff? Round your answers according to the rules for measurements.



The cliff is _____ meters tall.
Figure is not to scale.

26. From a window 28 feet above the ground, the angle of elevation to the top of another building is 34°. The distance between the buildings is 60 feet. Find the height of the building to the nearest tenth of a foot.
The height of the building is ____ feet.

27.



Identify the legs of the triangle involving θ, T, and the length of leg 1.
Opposite:

1. T
2. half of skip distance
3. leg 1 length

Adjacent:

1. T

2. half of skip distance

3. leg 1 length

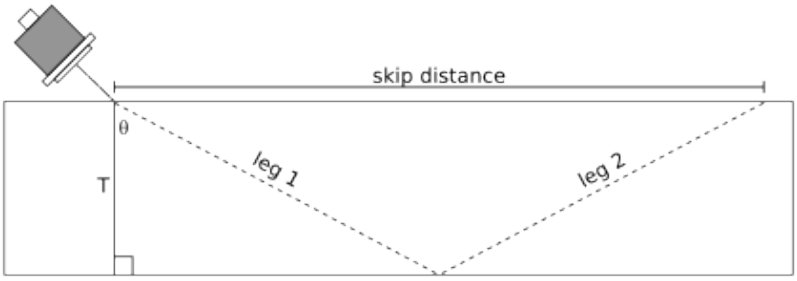
Hypotenuse:

1. T

2. half of skip distance

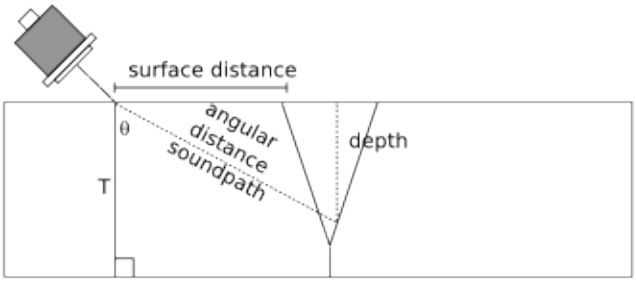
3. leg 1 length

28.



Write an expression for the following lengths in terms of T and θ.
Leg = _____
Leg path = _____ (path consisting of both legs)
Skip distance = _____

29.



Use a trig function to calculate the length of the following. s is the length of the surface distance.
depth = _____
soundpath = _____

30. Snell's Law about refraction applied to sound can be stated as $\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{V_1}{V_2}$.

$V_1 = 4600$ m/s in cast iron and $V_2 = 5300$ in glass. If an ultrasonic sound crosses from cast iron into glass at an angle of 45°. What angle from the normal will the sound make in the glass?

_____

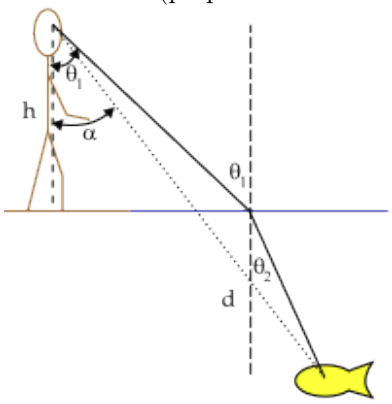
31. Snell's Law of Refraction models how waves (like light) bend when they pass from one substance into another. The model is

$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

where

n_1 and n_2 are indices of refraction for the two materials

θ_1 and θ_2 are the angles from the vertical (perpendicular to the surface)



In this situation the person is looking down (angle of declination) at 42°. Eye height is $h = 76$ inches, and the fish is $d = 25$ in below the surface. $n_1 = 1.00029$ (air) and $n_2 = 1.33$ (water).

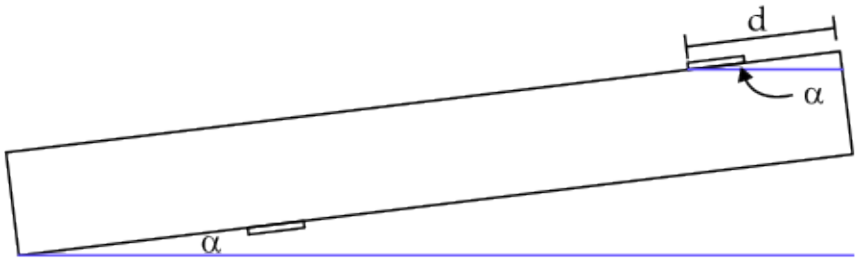
What is the angle θ_2? _____

What is the horizontal distance from the person to the point on the surface at which he is looking (between the vertical lines)? _____

What is the horizontal distance from that point on the surface (vertical line) to the fish? _____

What is the angle α that is the actual angle from the eye directly toward the fish? _____

32.



When a gas tank, which is a rectangular prism, is not level with the ground, a portion of the tank will not be able to be filled because it is higher than the hole through which the fuel comes. In the image above this is the area above the (blue) line at the top, right side of the tank.

Suppose this tank has the filler hole $d = 6$ in from the front edge of the tank, and the width of the tank is 36 in (this is the dimension not shown in the image). If the angle is $\alpha = 4°$, the unfillable portion has what volume in gallons? ____

There are 231 cubic inches per gallon. Round to the nearest hundreth of a gallon.

Quantitative Literacy. Use your knowledge about trigonometric functions and other math to answer these questions.

33. The beam from an ultrasonic testing probe spreads starting at the end of the near field. The angle of spread is given by $\sin\theta = 1.22\dfrac{\lambda}{D}$ where λ is the wavelength and D is the diameter of the probe (crystal generating the vibration).

A straighter beam (smaller angle) is desirable for testing. What conditions are best for a straight beam? Note that for angles between 0^@ and 90^@ if we want a small angle we also want a small sine value.

The wavelength should be

1. small
2. large

The frequency should be

1. small
2. large

The diameter should be

1. small
2. large

7.2 Inverse Trigonometric Relationships

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
- Analyze right triangles (skill)
- Identify properties of sine and cosine functions (skill)

The previous section presented relationships between side lengths and angles and presented solving problems where we know an angle. This section presents examples where we have side lengths and desire to know an angle.

7.2.1 Inverse Trigonometric Functions

The trigonometric functions presented above provide a side ratio given an angle. It is also possible to calculate the angle given a side ratio. We use the so called inverse trigonometric functions for this. There are two common notations for them which are shown in Table 7.2.1.

Table 7.2.1 Inverse Trigonometric Functions

Function	Inverse Function	
$\sin\alpha = r$	$\arcsin r = \alpha$	$\sin^{-1} r = \alpha$
$\cos\alpha = r$	$\arccos r = \alpha$	$\cos^{-1} r = \alpha$
$\tan\alpha = r$	$\arctan r = \alpha$	$\tan^{-1} r = \alpha$

Note the notation $\sin^{-1} x$ shows up on calculator keys and in many books. It is unfortunately easy to confuse with $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$. As a result that notation will not be used in this text.

Example 7.2.2 What is the measure of both non-right angles in Figure 7.1.8? Use technology to calculate.

We can use the arcsine function.

$$\alpha = \arcsin(3/\sqrt{34}) \approx 31^\circ.$$
$$\theta = \arcsin(5/\sqrt{34}) \approx 59^\circ$$

We can also use the arccosine function.

$$\alpha = \arccos(5/\sqrt{34}) \approx 31^\circ.$$
$$\theta = \arccos(3/\sqrt{34}) \approx 59^\circ$$

▲

Example 7.2.3 A right triangle has legs of lengths 4 and 8. What are the measures of the non-right angles?

Solution 1. Because we have the two legs, we can use the arctangent function to calculate the angles. The roles of adjacent and opposite switch for the two angles.

$$\arctan\left(\frac{4}{8}\right) \approx 26.57.$$
$$\arctan\left(\frac{8}{4}\right) \approx 63.43.$$

Solution 2. Because we have two legs, we can use the Pythagorean Theorem to calculate the third side length, then use arcsine.

$$4^2 + 8^2 = c^2.$$
$$80 = c^2.$$
$$8.944 \approx c.$$

$$\arcsin\left(\frac{4}{8.944}\right) \approx 26.57.$$
$$\arcsin\left(\frac{8}{8.944}\right) \approx 63.44.$$

Notice that the larger angle is slightly different from the first solution. This is the result of using the approximate hypotenuse. ▲

Checkpoint 7.2.4 If a right triangle has legs of length $a = 11$ and $b = 23$, what are the angles in degrees
opposite from side of length $a = 11$: $A \approx$ ___°
opposite from side of length $b = 23$: $B \approx$ ___°
Round to units.

7.2.2 Solving Triangles

With inverse trigonometric functions we can calculate all side lengths and all angles starting with just side lengths.

Example 7.2.5 A right triangle has a leg of length 12 and the hypotenuse has length 13, what is the length of the other leg? What are the measures of the angles?

We know lengths of two sides so we can apply the Pythagorean Theorem to calculate the length of the third side.

$$12^2 + b^2 = 13^2.$$
$$b^2 = 13^2 - 12^2.$$
$$b^2 = 25.$$
$$b = 5.$$

For the angle opposite the side of length 12, we can use the inverse sine to calculate the angle.

$$\arcsin\left(\frac{12}{13}\right) \approx 67°.$$

For the angle opposite the side of length 5, we can now use the angle sum theorem.

$$90° + 67° + B \approx 180°.$$
$$B \approx 180° - 90° - 67°.$$
$$B \approx 23°.$$

The angle measure is approximate, because we rounded the result of the inverse sine calculation. Rounding to units here was arbitrarily chosen because we do not have a context. ▲

Example 7.2.6 A right triangle has legs of length 14 and 48.

(a) What is the length of the other side?

Solution. The other side is the hypotenuse. We can calculate it using the Pythagorean theorem.

$$\begin{aligned} c^2 &= 14^2 + 48^2 \\ &= 2500. \\ c &= \sqrt{2500} \\ &= 50. \end{aligned}$$

(b) What are the measures of the angles?

Solution. For the angle opposite the side of length 14 because we know the lengths of both legs, we can use

$$\arctan\left(\frac{14}{48}\right) \approx 16^\circ.$$

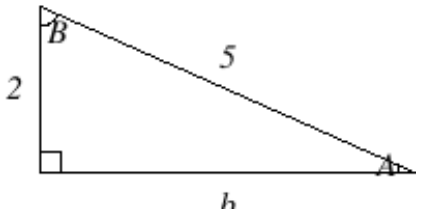
We could have used arcsine and the length of the hypotenuse, however, that length has rounding error which could affect this calculation.

For the other angle we can use the angle sum theorem.

$$\begin{aligned} 90^\circ + 16^\circ + B &\approx 180^\circ. \\ B &\approx 180^\circ - 90^\circ - 16^\circ. \\ B &\approx 74^\circ. \end{aligned}$$

▲

Checkpoint 7.2.7

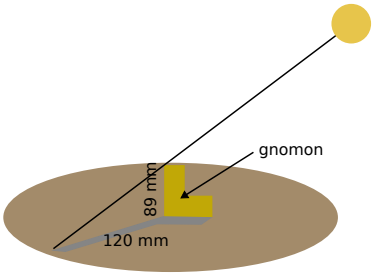


Suppose a = 2 and c = 5. Calculate:
$b \approx$ ____
$A \approx$ ____ degrees
$B \approx$ ____ degrees
Round all answers to one decimal place. Give angles in *degrees*.

7.2.3 Calculating angles using trig functions

This section demonstrates how to calculate the angles when we know the side lengths of a right triangle.

Example 7.2.8 If the gnomon of a sundial (L shaped piece that casts a shadow) is 89 mm tall, and the shadow the sun casts on the sundial is $12\bar{0}$ mm long, what is the angle of elevation of the sun? These are measurements, so use significant digits.



The gnomon and surface form the legs of a right triangle. This means we can use the inverse tangent to calculate the angle. With respect to the angle of the sun, the surface is the adjacent and the gnomon is the opposite.

$$\begin{aligned}\tan(\theta) &= \frac{89}{12\bar{0}}.\\ \theta &= \arctan\left(\frac{89}{12\bar{0}}\right)\\ &\approx \arctan(0.7\underline{4}166667)\\ &\approx 3\underline{6}.563096\\ &\approx 37^\circ.\end{aligned}$$

▲

Example 7.2.9 An airliner is at 13,000 feet MSL and is cleared to descend to 9,000 feet MSL. This descent will be accomplished over 22 nm. What is the angle of descent? Round to units, because aircraft instruments are not more precise.

First, we will need to convert 22 nm to feet. The unit conversions in Table 1.1.2 suggest we can multiply $\frac{6076 \text{ ft}}{1 \text{ nm}} \cdot 22 \text{ nm} = 133672 \text{ ft}$.

The 133,672 ft is the length of the adjacent side. The length of the opposite side is the change in altitude which is 13,000−9,000 = 4,000 ft. Because the horizontal and vertical components are legs of a right triangle, we can use the inverse tangent function to calculate the angle. The horizontal change is the adjacent side, and the vertical change is the opposite side.

$$\begin{aligned}\tan(\theta) &= \frac{4000}{133672}.\\ \theta &= \arctan\left(\frac{4000}{133672}\right)\\ &\approx \arctan(0.02992399)\\ &\approx 1.71400703\\ &\approx 2^\circ.\end{aligned}$$

▲

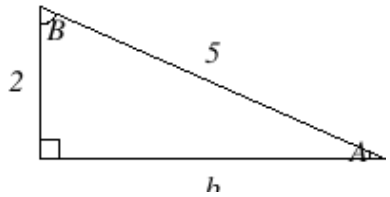
Checkpoint 7.2.10 An aircraft is at 20,000 feet MSL and is cleared to descend to 5,000 feet MSL. This descent will be accomplished over 20 nm. What is the angle of descent? ___

Round to the nearest unit.

7.2.4 Exercises

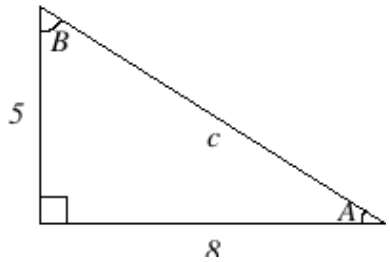
Solving Triangles. Calculate side lengths and angles using trigonometric functions.

1.



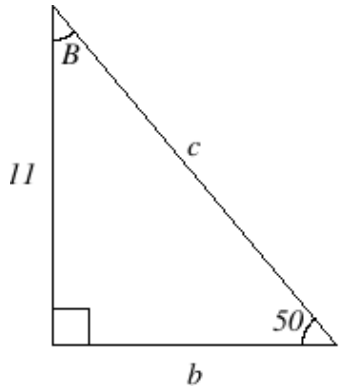
Suppose a = 2 and c = 5. Calculate:
$b \approx$ ____
$A \approx$ ____ degrees
$B \approx$ ____ degrees
Round all answers to one decimal place. Give angles in *degrees*.

2.



Suppose a = 5 and b = 8. Calculate:
c = _____
A = _____ degrees
B = _____ degrees
Round all answers to one decimal place. Give angles in *degrees*.

3.



Suppose a = 11 and A = 50 degrees. Calculate:
$b \approx$ ____
$c \approx$ ____

$B \approx$ ____ degrees

Round all answers to one decimal place. Give angles in *degrees*

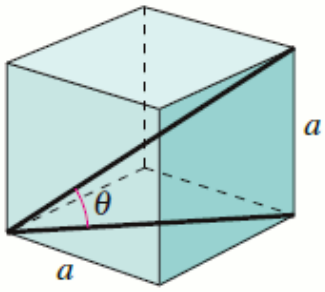
Applications. Use inverse trigonometric functions to answer these questions.

4. The optimal angle for an extension ladder is 75°. This is approximated by placing the feet of the ladder 1/4 the length of the ladder from where the rails rest at the top.

Suppose the ladder is 140 inches long. How far from the edge should the feet of the ladder be placed? _____

What angle does this actually make? _____

5. What is the angle , to the nearest tenth of a degree, between a diagonal of a cube and a diagonal of a face of that cube.



angle = _____ °

Report answer accurate to 1 decimal places.

6. You are skiing down a mountain with a vertical height of 1282 feet. The distance on the surface of the mountain, from the top to the base, is 3076.8 feet. What is the angle of elevation from the base to the top of the mountain?

Express your answer rounded to the nearest degree.

_____ degrees

7. A 36-foot tree casts a 16-foot shadow. Find the measure of the angle of elevation to the sun to the nearest degree.

Angle of elevation is ____ degrees.

8. A plane flying at an altitude of 16,000 feet begins descending when the end of the runway is 52,000 feet from a point on the ground directly below the plane. Find the measure of the angle of descent (depression) to the nearest degree.

The angle of descent is ____ degrees.

Quantitative Literacy. Use inverse trigonometric functions to answer these practical questions.

9. The optimal angle for an extension ladder is 75°. This angle is measured from the middle of the rails (center of a foot). The angle is approximated by placing the feet of the ladder 1/4 the length of the ladder from where the rails rest at the top. The foot of a ladder is about 4" long. When placing the ladder the a person might measure the distance to either edge or the middle of the feet. We ask if this makes a significant difference in the angle.

Suppose the ladder is 252 inches long. For each of the following use the distance from the edge to the middle of the feet to calculate the angle.

If the distance is measured to the middle of the foot of the ladder, what is the angle? _____

If the distance is measured to the part of the foot closest to the wall (so the middle is 2 inches further away from the wall), what is the angle? _____

If the distance is measured to the part of the foot farthest from the wall (so the middle is 2 inches closer to the wall), what is the angle? _____

Does the difference in angle exceed 2° (which would not have a significant safety impact)?

1. no

2. yes

7.3 Non-Right Triangles

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
- Analyze non-right triangles (skill)

In Section 7.1 we learned about relationships between angles of the triangles and their sides. However, most of our work was restricted to right triangles. This section demonstrates how to make similar calculations on non-right triangles.

The one relationship that did not require a right angle is the Triangle Angle Sum Theorem.

7.3.1 Law of Sines

In Section 7.1 we saw that there was a relationship between angles of a triangle and the side ratios. More generally there is a relationship between the magnitude of an angle of a triangle and the length of the side opposite it. The following theorem expresses this relationship.

Theorem 7.3.1 Law of Sines. *For a triangle with angles and sides as labeled in Figure 7.3.2,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

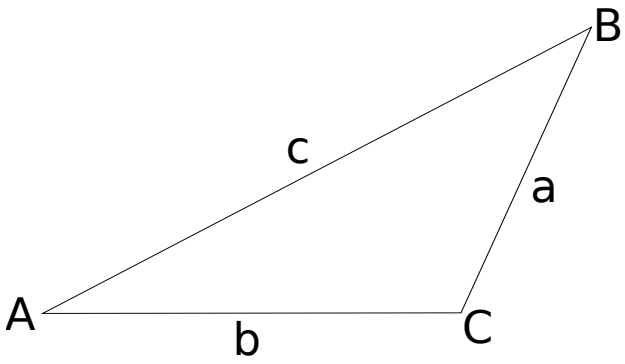


Figure 7.3.2 Labeled Triangle

Example 7.3.3 A triangle has an angle with measure 50° which is opposite a side of length 6. The triangle has another angle with measure 45°.

Rounding in this example is arbitrarily chosen to be two decimal places.

(a) What is the length of the side opposite the 45° angle?

Because we know two angles and a side opposite one of them we can use the law of sines.

$$\frac{\sin(50°)}{6} = \frac{\sin(45°)}{b}. \qquad \text{Set up Law of Sines}$$

$$b\sin(50°) = 6\sin(45°). \qquad \text{Clear the denominators}$$

$$b = 6\frac{\sin(45°)}{\sin(50°)}. \qquad \text{Divide to solve for } b$$
$$\approx 5.54.$$

(b) What is the third angle and the length of the side opposite it?

We can use the angle sum fact to calculate the third angle. $50° + 45° + \alpha = 180°$ so $\alpha = 85°$. As in the previous step we know an angle (50°), the side opposite it (6), and another angle (85°) so we can use the Law of Sines.

$$\frac{\sin(50°)}{6} = \frac{\sin(85°)}{c}. \qquad \text{Set up the Law of Sines}$$
$$c\sin(50°) = 6\sin(85°). \qquad \text{Clear the denominators}$$
$$c = 6\frac{\sin(85°)}{\sin(50°)} \qquad \text{Divide to solve for } c$$
$$\approx 7.80.$$

If you recall triangle congruency, this was known as Angle-Angle-Side. ▲

Example 7.3.4 A triangle has two angles with measure 40° and 60.20°. The side between these two angles has length 7.66.

What is the measure of the other angle, and what are the other side lengths? Everything will be rounded to two decimal places.

The third angle measure is the easiest to calculate, because we can use the angle sum theorem. $40° + 60.20° + \alpha = 180°$. Thus $\alpha = 79.80°$.

Now we know an angle (α) and the length of the side opposite it (7.66). This enables us to use the Law of Sines to calculate the other two side lengths.

$$\frac{\sin(40°)}{a} = \frac{\sin(79.80°)}{7.66}. \qquad \text{Set up Law of Sines}$$
$$\frac{a}{\sin(40°)} = \frac{7.66}{\sin(79.80°)}. \qquad \text{Proportion true both ways}$$
$$a = \frac{7.66\sin(40°)}{\sin(79.80°)} \qquad \text{Multiply to solve}$$
$$\approx 5.00281961$$
$$\approx 5.00.$$

We calculate the third side the same way.

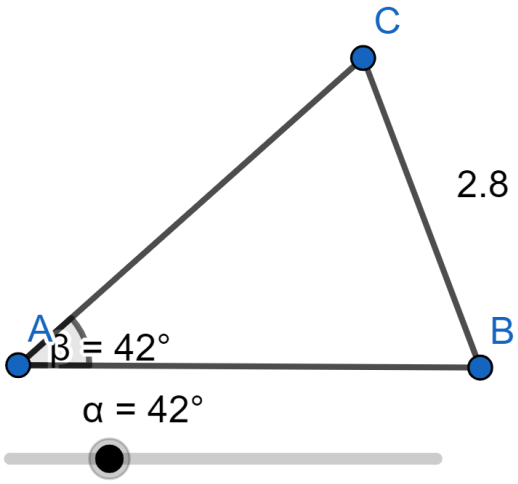
$$\frac{\sin(60.20°)}{a} = \frac{\sin(79.80°)}{7.66}.$$
$$\frac{a}{\sin(60.20°)} = \frac{7.66}{\sin(79.80°)}.$$
$$a = \frac{7.66\sin(60.20°)}{\sin(79.80°)}$$
$$\approx 6.75382345$$
$$\approx 6.75.$$

For triangle geometry, this was known as Angle-Side-Angle. ▲

Checkpoint 7.3.5 A triangle has angles of measure $A = 30°$ and $B = 60°$. The side opposite angle A has length 6.0. What is the length of the side opposite angle B? _____

Round to the nearest tenth.

The following activity demonstrates the relationship between the magnitude of an angle and the length of the side opposite that angle.



Standalone
In Context
Embed

Figure 7.3.6 Experiment to Demonstrate the Angle/Opposite Side Relationship

Activity 7.3.7 Use the illustration above to answer the questions below.

(a) Use the slider to set the angle to about 10°.

What happens to the length of the opposite side as you increase the angle to 170°?

(b) Recall the Triangle Angle Sum relationship. As the angle at A increases what must be happening to the sum of the measures of the other two angles?

What must be happening to each of the other two angles?

(c) In this illustration the side connecting A and B is remaining the same length. The angle opposite that side is changing as you noted in the previous step.

Combining the results of the previous two steps, what do you think is true about the following. The angle opposite a larger side is bigger/smaller/unrelated to the angle opposite a shorter side.

7.3.2 Ambiguous Triangles

Above we calculated angles and side lengths given partial information about a triangle (two angles and a side). This section presents a case (two sides and an angle) that we cannot resolve without additional information.

Example 7.3.8 A triangle has an angle of measure 45° and the side opposite it is length 4. Another side has length 5. What are the other angles and side lengths?

We can try to use the Law of Sines.

$$\frac{\sin(45°)}{4} = \frac{\sin(\theta)}{5}.$$

$$5 \cdot \frac{\sin(45°)}{4} = \sin(\theta).$$

$$\arcsin\left(\frac{5}{4}\sin(45°)\right) = \theta.$$

$$62.11° \approx \theta.$$

Using the triangle angle sum theorem we learn the other angle measure. $45°+62.11°+\alpha = 180°$ or $\alpha \approx 72.89°$. We use the Law of Sines again to calculate the length of the final side.

$$\begin{aligned}\frac{\sin(45°)}{4} &= \frac{\sin(72.89°)}{c}.\\ \frac{4}{\sin(45°)} &= \frac{c}{\sin(72.89°)}.\\ c &= 4\frac{\sin(72.89°)}{\sin(45°)}.\\ c &\approx 5.41.\end{aligned}$$

This gives us a triangle with angles: 45°, 62.11°, and 72.89°; and with side lengths: 4, 5, and 5.41.

However, $\sin(117.89°) = \sin(62.11°)$, that is, $\arcsin\left(\frac{5}{4}\sin(45°)\right)$ had multiple possible angles. We will repeat the calulcations above using 117.89° as the second angle. The third angle is $45° + 117.89° + \alpha = 180°$ or $\alpha \approx 17.11°$.

$$\begin{aligned}\frac{\sin(45°)}{4} &= \frac{\sin(17.11°)}{c}.\\ c\sin(45°) &= 4\sin(17.11°).\\ c &= 4\frac{\sin(17.11°)}{\sin(45°)}.\\ c &\approx 1.66.\end{aligned}$$

Notice we have two, distinct triangles that match the initial angle and side information. They can be seen in Figure 7.3.9. This indicates an ambiguity if all we know is this particular information.

If you recall triangle geometry, this was the case Side-Side-Angle which does not prove congruent triangles. ▲

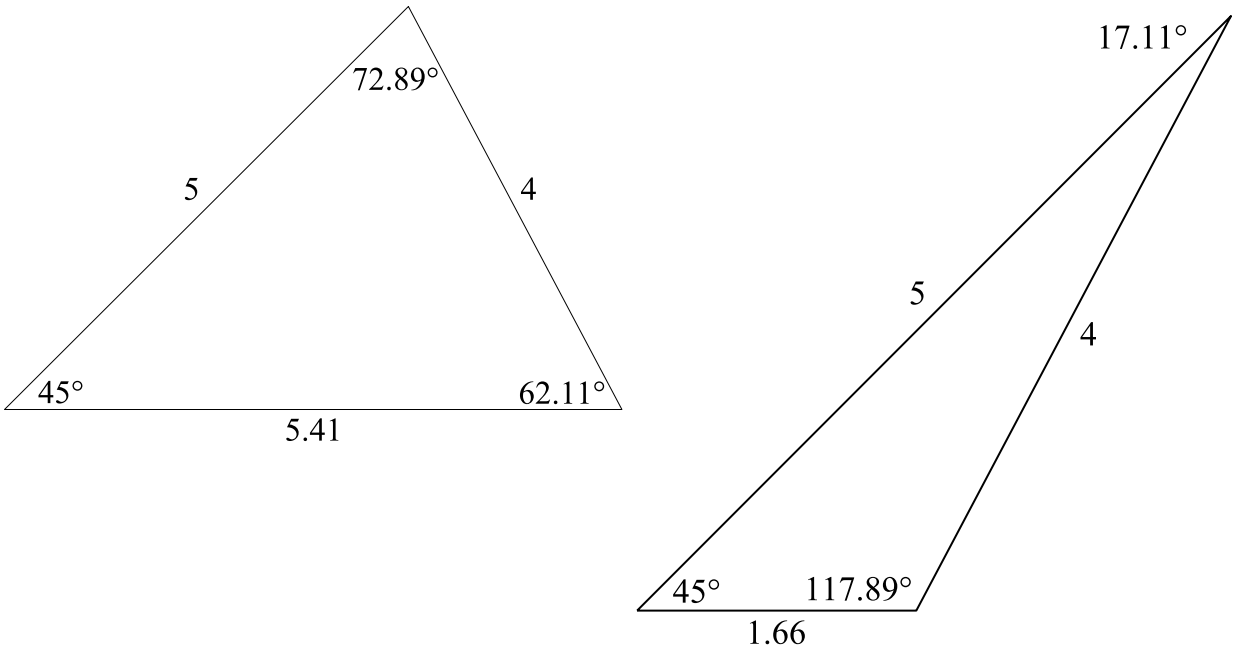


Figure 7.3.9 Two Possible Triangles

When using the Law of Sines we will need to restrict ourselves to the two cases for which it works: Angle-Angle-Side and Angle-Side-Angle.

7.3.3 Law of Cosines

For right triangles we know the Pythagorean theorem is a relationship between the sides of those triangles. For triangles without a right angle that relationship must be slightly modified. The more general statement is below.

Theorem 7.3.10 Law of Cosines. *For any triangle with side lengths a, b, c and angle C which is opposite the side with length c*

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

Example 7.3.11 A triangle has sides of lengths 4.00, 5.39, and 6.13. What are the angles?

We can use the Law of Cosines.

$$\begin{aligned}
4^2 &= 5.39^2 + 6.13^2 - 2(5.39)(6.13)\cos(A).\\
16 &= 29.0521 + 37.5769 - 66.0814\cos(A).\\
-50.629 &= -66.0814\cos(A).\\
\frac{-50.629}{-66.0814} &= \cos(A).\\
0.76616113 &\approx \cos(A).\\
\arccos(0.76616113) &\approx A.\\
39.98959796^\circ &\approx A.\\
40.0^\circ &\approx A.
\end{aligned}$$

With an angle, we could now use the Law of Sines, but for practice we will use the Law of Cosines again.

$$\begin{aligned}
5.39^2 &= 4^2 + 6.13^2 - 2(4)(6.13)\cos(B).\\
29.0521 &= 16 + 37.5769 - 49.04\cos(B).\\
-24.5248 &= -49.04\cos(B).\\
0.50009788 &\approx \cos(B).\\
\arccos(0.50009788) &\approx B.\\
59.99352413^\circ &\approx B.\\
60.0^\circ &\approx B.
\end{aligned}$$

Knowing two of the angles we can use the triangle angle sum theorem to calculate the third angle measure. $40.0^\circ + 60.0^\circ + C = 180^\circ$ so $C = 80.0^\circ$.

For triangle congruency, this was known as Side-Side-Side. ▲

Example 7.3.12 A triangle has sides with lengths 5 and 7 and the angle between them is 40°. What are the length of the other side and the measures of the other angles?

Because we have two sides (a, b) and the angle between them (C) we can use the Law of Cosines.

$$\begin{aligned}
c^2 &= 5^2 + 7^2 - 2(5)(7)\cos(40^\circ)\\
c^2 &\approx 20.37688900.\\
\sqrt{c^2} &\approx \sqrt{20.37688900}.\\
c &\approx 4.514076761\\
&\approx 4.51.
\end{aligned}$$

Now that we know a side and the angle opposite it, we can use the Law of Sines to calculate the remaining two angles.

$$\begin{aligned}
\frac{\sin(40^\circ)}{4.514076761} &= \frac{\sin(A)}{5}.\\
0.71198126 &= \sin(A).
\end{aligned}$$

$$\arcsin(0.71198126) = A.$$
$$45.39634537^\circ \approx A.$$
$$45.4^\circ \approx A.$$

Finally we can use the triangle angle sum theorem to calculate the final angle. $40^\circ + 45.4^\circ + B = 180^\circ$ so $B = 94.6^\circ$.

For triangle congruency this was known as Side-Angle-Side. ▲

Checkpoint 7.3.13 A triangle has sides with lengths $a = 6.00$, $b = 9.19$, and $c = 8.08$. Calculate all of the angles of this triangle.

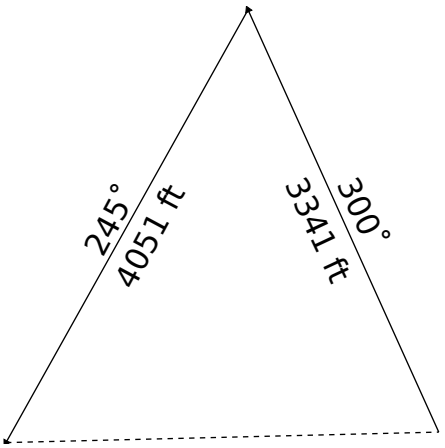
$A \approx$ ____ (across from side of length $a = 6.00$)
$B \approx$ ____ (across from side of length $b = 9.19$)
$C \approx$ ____ (across from side of length $c = 8.08$)
Round to two decimal places.

7.3.4 Using Trigonometric Laws in Applications

When we encounter non-right triangles in applications, we will need to check if we have the information needed to use the Law of Sines or the Law of Cosines.

Example 7.3.14 A sailboat sails 3341 ft at heading 300° then turns to heading 245° and sails 4051 ft. How far is the sailbot from its starting position? Round to units, because fractions of a foot are not meaningful in the motion of a vehicle.

First, it helps to sketch an image.



In our sketch we see a non-right triangle. We know the lengths of two sides, and we want the length of the third side (distance from original location). From the headings we can calculate the angle between the two sides of known length. This provides side-angle-side which allows us to use the Law of Cosines to calculate the desired distance (length of third side).

We are given the headings for two segments but not an angle. The angle representing the amount of the turn is the change in headings. If we were at 300° and ended up at 245° then we turned left $300^\circ - 245^\circ = 55^\circ$.

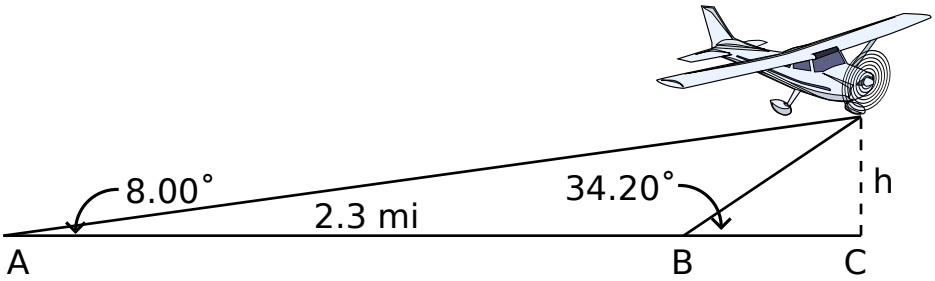
Now we can apply the Law of Cosines.

$$\begin{aligned} c^2 &= 3341^2 + 4051^2 - 2(3341)(4051)\cos(55^\circ) \\ &= 27572882 - 27068782\cos(55^\circ) \\ &\approx 27572882 - 15526015.51 \\ &\approx 12046866.49. \\ c &\approx \sqrt{12046866.49} \\ &\approx 3470.859618 \\ &\approx 3471. \end{aligned}$$

Thus the sailboat ended up 3471 ft from its starting point. ▲

Example 7.3.15 Two observers are 2.3 miles apart on the ground at the same altitude. At the same time they record the angle at which they saw an aircraft pass. The observer at point A recorded an angle of 8.00° from horizontal. The observer at point B recorded an angle of 34.20° from horizontal. How high above ground was the aircraft?

First, it helps to sketch an image.



How do we know that the larger angle is closer to the aircraft? If we think about an aircraft flying toward us, we realize that our head tips up (bigger angle) as it becomes closer. Thus the closer observer will have the larger angle.

In our sketch we see multiple triangles including some right triangles. We want to calculate the height of the aircraft above ground which is part of the small, right triangle with vertices B, C, and the airplane. Initially we know only the bottom angles (34.20° and the right angle) and none of the lengths. If we can calculate the length of the side from B to the airplane (on the left), then we could use the angles and that side length with the Law of Sines to calculate the height.

That left side is also part of a non-right triangle with vertices at points A, B, and the aircraft. The side from A to B has length 2.9 mi, and we know the angles on either side. This means we have angle-side-angle information, so we can apply the Law of Sines to calculate the length of the hypotenuse we need. In particular

$$\frac{\sin(P)}{2.3} = \frac{\sin(8.00^\circ)}{c}$$

To calculate the angle at the plane (labeled P above) we will need to use the Triangle Angle Sum theorem. However, the angle we are given is the other side of the non-right triangle. The angle we need now is on the other side of that line, so its measure is $180^\circ - 34.20^\circ = 145.80^\circ = B$. Now, b $8.00^\circ + 145.80^\circ + P = 180^\circ$ So, the angle at the aircraft is $P = 26.20^\circ$.

Now we can apply the Law of Sines.

$$\begin{aligned}
\frac{\sin(8.00^\circ)}{c} &= \frac{\sin(26.20^\circ)}{2.3 \text{ mi}}. \\
\frac{c}{\sin(8.00^\circ)} &= \frac{2.3 \text{ mi}}{\sin(26.20^\circ)}. \\
c &= \frac{(2.3 \text{ mi})\sin(8.00^\circ)}{\sin(26.20^\circ)} \\
&\approx 0.72501447 \text{ mi} \\
&\approx 0.73 \text{ mi}.
\end{aligned}$$

We now know the length of the hypotenuse of the right triangle whose leg is the desired aircraft altitude. This gives us angle-angle-side (first angle is the right angle). This means we can use the Law of Sines to calculate the height.

$$\begin{aligned}
\frac{\sin(34.20^\circ)}{h} &= \frac{\sin(90^\circ)}{0.72501447 \text{ mi}}. \\
\frac{h}{\sin(34.20^\circ)} &= \frac{0.72501447 \text{ mi}}{\sin(90^\circ)}.
\end{aligned}$$

$$\begin{aligned} h &= \frac{(0.72501447 \text{ mi}) \sin(34.20°)}{\sin(90°)} \\ &\approx 0.40751858 28 \text{ mi} \\ &\approx 0.41 \text{ mi}. \end{aligned}$$

This is in miles. It will be easier to interpret in feet. The conversion ratio in Table 1.1.2 suggests we can multiply

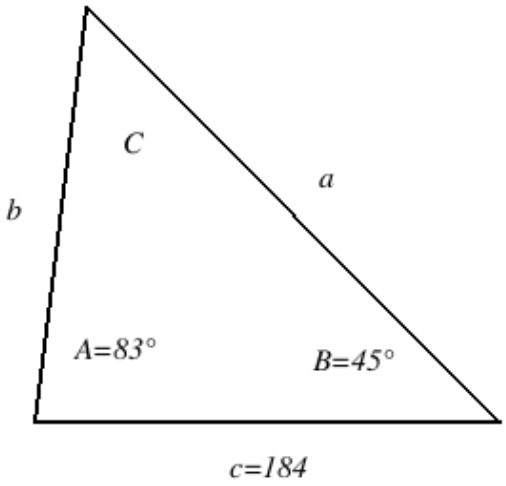
$$(0.40751862 \text{ mi}) \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \approx 2151.6983 \text{ ft} \approx 2200 \text{ ft}.$$

▲

7.3.5 Exercises

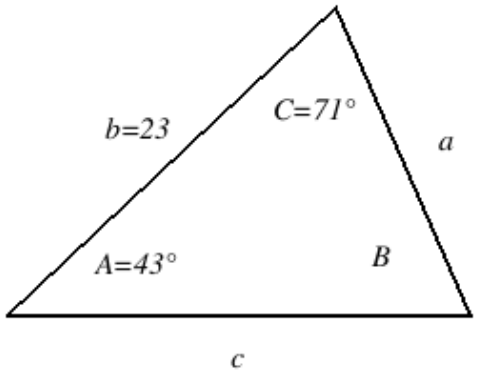
Solving Triangles. Use the trigonometric laws to calculate side lengths and angles.

1.



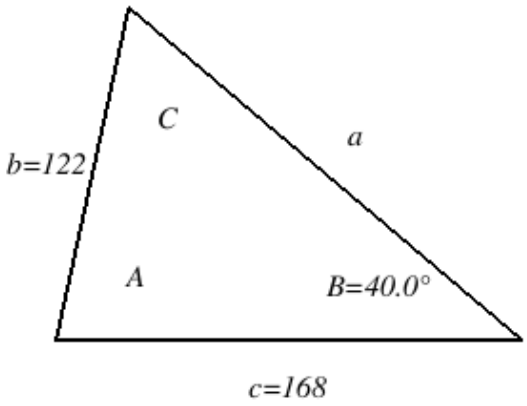
$b =$ _____
Round to 2 decimal places.

2. Solve the triangle



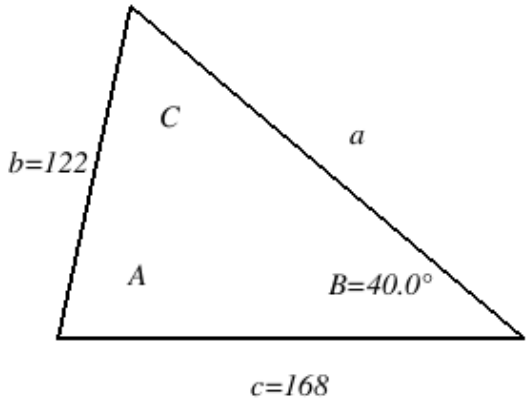
$\angle B$ is _____ degrees
$a =$ _____
$c =$ _____
Round to 2 decimal places.

3.



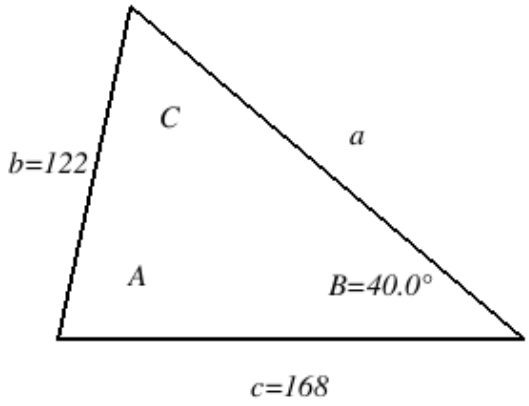
$a =$ _____
$\angle A =$ _____ degrees
$\angle C =$ _____ degrees
Use significant figures.

4.



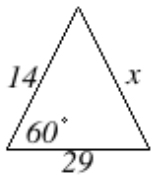
$a =$ _____
$\angle A =$ _____ degrees
$\angle C =$ _____ degrees
Use significant figures.

5.



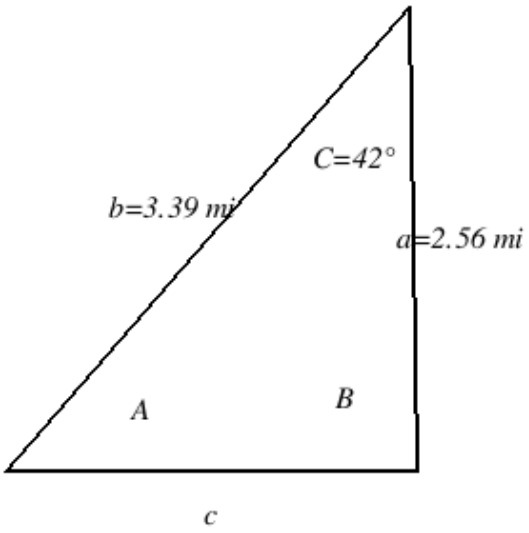
$a =$ _____
$\angle A =$ _____ degrees
$\angle C =$ _____ degrees
Use significant figures.

6. Given the triangle



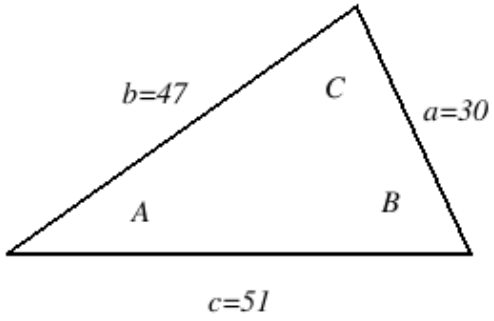
$xx =$ _____

7.



$c =$ _____ mi
Round to 2 decimal places.

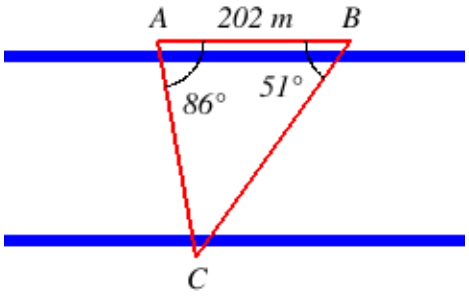
8.



Solve the triangle
$A =$ _____°
$B =$ _____°
$C =$ _____°
Round to 2 decimal places.

Applications. Identify triangles in these applications and use trigonometric laws to answer these questions.

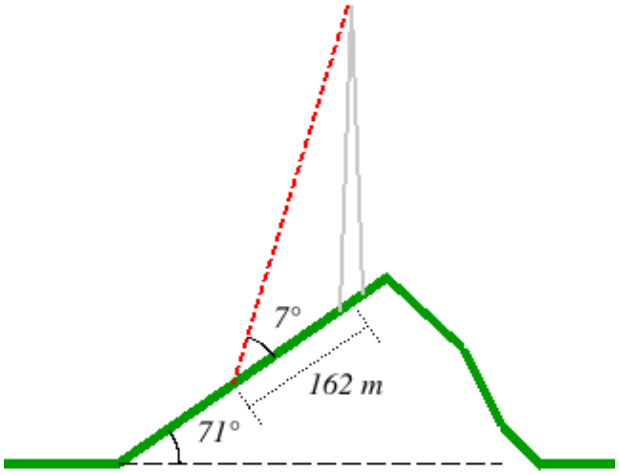
9. To find the distance across a river, a surveyor choose points A and B, which are 202 m apart on one side of the river. She then chooses a reference point C on the opposite side of the river and finds that $\angle BAC \approx 86$ and $\angle ABC \approx 51$.



*NOTE: The picture is NOT drawn to scale.*Approximate the distance from point A to point C.
distance = _____ m
Find the distance across the river.
height = _____ m
Enter your answer as a number; your answer should be accurate to 2 decimal places.

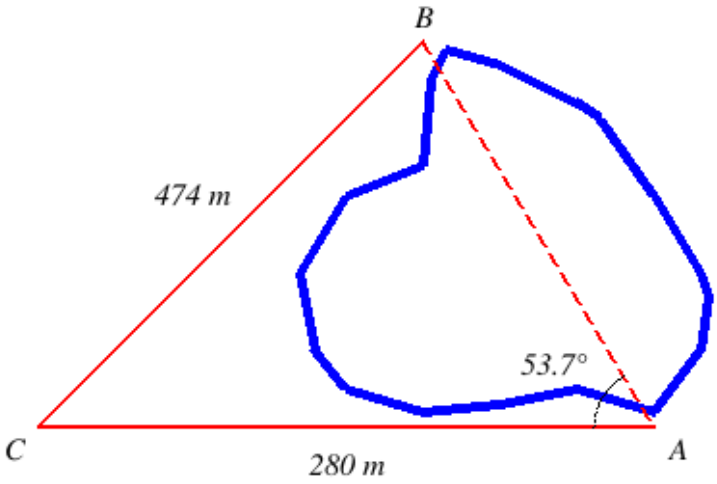
10. A communications tower is located at the top of a steep hill, as shown. The angle of inclination of the hill is 71°. A guy wire is to be attached to the top of the tower and to the ground, 162 m downhill from the base of the tower. The angle formed by the guy wire is 7°. Find the length of

the cable required for the guy wire.



*NOTE: The picture is NOT drawn to scale.*length of guy-wire = _____ m
Enter your answer as a number; your answer should be accurate to 2 decimal places.

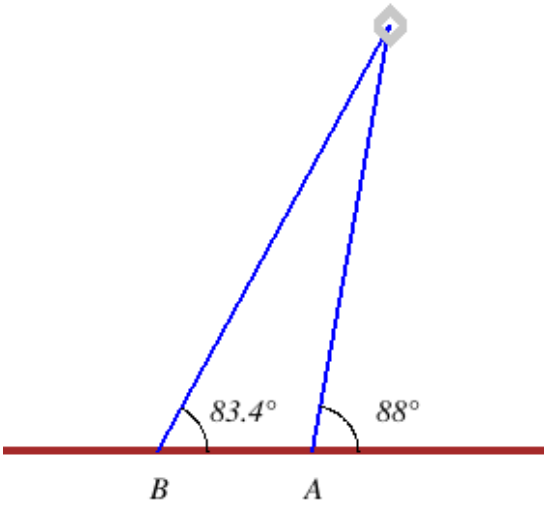
11. Points A and B are separated by a lake. To find the distance between them, a surveyor locates a point C on land such than $\angle CAB = 53.7$. Find the distance across the lake from A to B.



*NOTE: The triangle is NOT drawn to scale.*distance = _____ m
Enter your answer as a number; your answer should be accurate to 2 decimal places.

12. The path of a satellite orbiting the earth causes it to pass directly over two tracking stations A and B, which are 52 km apart. When the satellite is on one side of the two stations, the angles of

elevation at A and B are measured to be 88° and 83.4°, respectively.

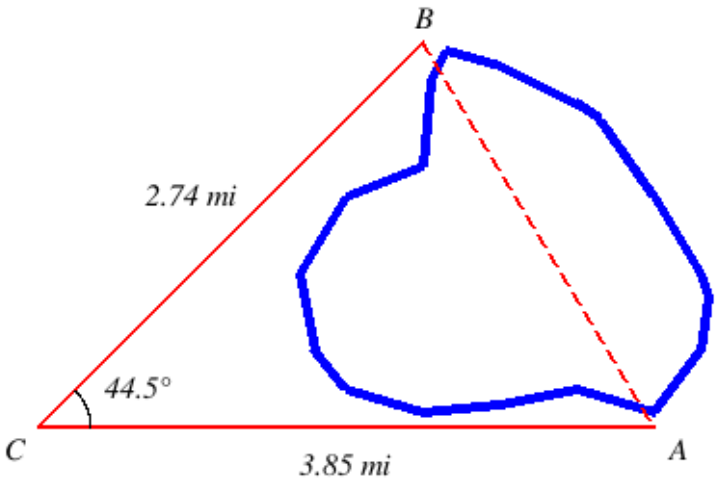


*NOTE: The picture is NOT drawn to scale.*How far is the satellite from station A?
distance from A = _____ km
How high is the satellite above the ground?
height = _____ km
Enter your answer as a number; your answer should be accurate to 2 decimal places.

13. A pilot flies in a straight path for 1 h 30 min. She then makes a course correction, heading 10 degrees to the right of her original course, and flies 2 h in the new direction. If she maintains a constant speed of 685 mi/h, how far is she from her starting position?
Your answer is _____ mi;

14. A steep mountain is inclined 74 degree to the horizontal and rises to a height of 3400 ft above the surrounding plain. A cable car is to be installed running to the top of the mountain from a point 840 ft out in the plain from the base of the mountain. Find the shortest length of cable needed.
Your answer is _____ ft

15. To find the distance across a small lake, a surveyor has taken the measurements shown. Find the distance across the lake using this information.



*NOTE: The triangle is NOT drawn to scale.*distance = _____
Enter your answer as a number; your answer should be accurate to 2 decimal places.

16. The four sequential sides of a quadrilateral have lengths $a = 4.9$, $b = 7.7$, $c = 9.2$, and $d = 11.2$ (all measured in yards). The angle between the two smallest sides is $\alpha = 101$.

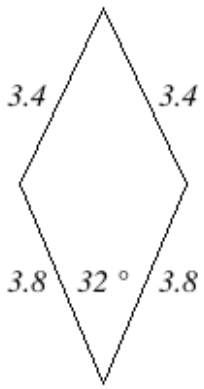
What is the area of this quadrilateral?

area = _____ yd²

Round to 2 decimal places.

17. A *kite* is a quadrilateral with two pairs of adjacent sides of equal length (think of simple ones you can fly).

A kite has two sides of length 3.8 and two sides of length 3.4 (all measured in yards). The angle between the sides of length 3.8 is 32°.



What is the area of this kite? _____ yd²

Round using measurement rules.

18. A surveyor starting from a point A moves N. 27°41 E. a distance of 599 11.5 to point B. Next, she moves N. 54°32 E. a distance of 477 1.75 to point C. Next, she walks S. 5°33 W. a distance of 1837 7.75 to point D. Finally, she returns to the starting point.

What distance must she walk to return to the starting point? (Answer accurate to the nearest quarter inch.)

_____ feet &; _____ inches

What heading does she walk from the fourth point to return to the starting point? (Answer accurate to the nearest minute.)

N. _____° _____ W.

What is the acreage of this plot of land?

area = _____ acre

Your answer should be accurate to 3 decimal places.

Notes:

∗ the prime symbol in an angle represents minutes: 60 = 1°

∗ the prime symbol in a length represents feet; double prime = inches;

∗ 1 acre = 4840 yd²

7.4 Sine Wave Properties

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
- Identify properties of sine and cosine functions (skill)

We began by looking at trigonometric functions in the context of triangles where they represent the ratio of side lengths. Here we will consider trigonometric functions in the context of properties of their graphs. The graphs have direct applications.

7.4.1 Beyond Triangles

This section illustrates how trigonometric functions can be defined on angles greater than 180°. In triangles every angle had to be less than 180° because the sum of the angles of a triangle are only 180°. However, in many applications rather than measuring angles on objects (like triangles), we are measuring how far or how many times around something has moved. For example a wheel on a car moves more than 180° when we are driving. Use Activity 7.4.1 to explore this idea.

Activity 7.4.1 Use the the illustration in Figure 7.4.2 to see how angles, including those bigger than 180°, are measured and how the trig functions act on these angles.

(a) Angles of measure 30° and 210°

(i) What is the sine value for both angles?

(ii) Compare the x coordinates of these two points on the sine graph.

(iii) Where is the triangle created by the angle 210°?

(b) Angles of measure 45° and 315°

(i) Compare the sine values for these two points.

(ii) Where is the triangle created by the angle 315°?

(c) Angles of measure 45° and 405°

(i) Compare the sine values for these two angles.

(ii) Where is the triangle created by the angle 405°?

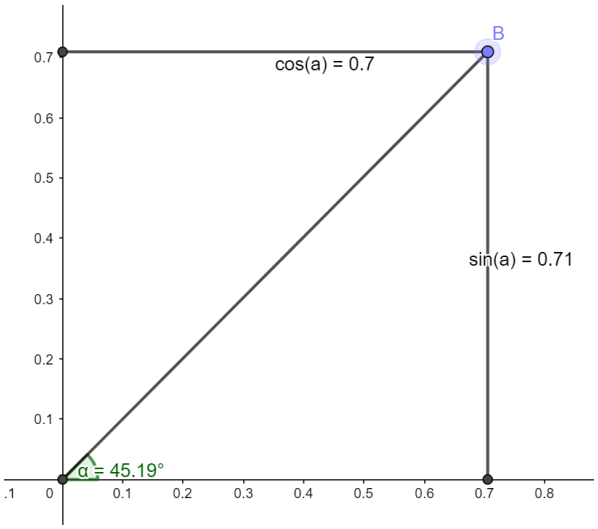
(iii) Note the angle displayed at the origin for 405°: why does it not match the slider angle?

(d) Angles of measure -45° and 315°

(i) Where are the triangles for these two points?

(ii) Move the slider from 0° to -45°. Which direction does the point move?

(iii) Note the angle displayed at the origin: explain why it is reasonable.



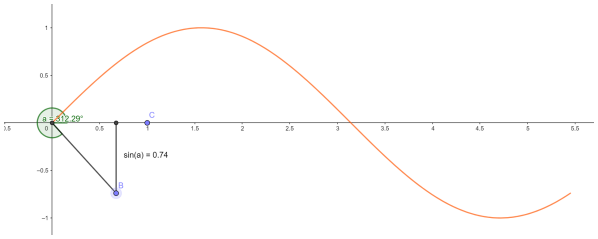
Standalone
In Context
Embed

Figure 7.4.2 Bigger Angles

Checkpoint 7.4.3 The angle 915° has the same sine value as which of these angles?

1. 15
2. 165
3. 195
4. 345

We can use the definition of sine as a ratio and this understanding of angles to produce a graph. In Figure 7.4.4 drag the slider until you have the full graph. A graph that extends over a longer range (and labeled in degrees) is in Figure 7.4.5.



Standalone
In Context
Embed

Figure 7.4.4 Produce Graph of Sine

Now we know what the graph of the sine function looks like and why it looks that way.

7.4.2 Properties of Sine Waves

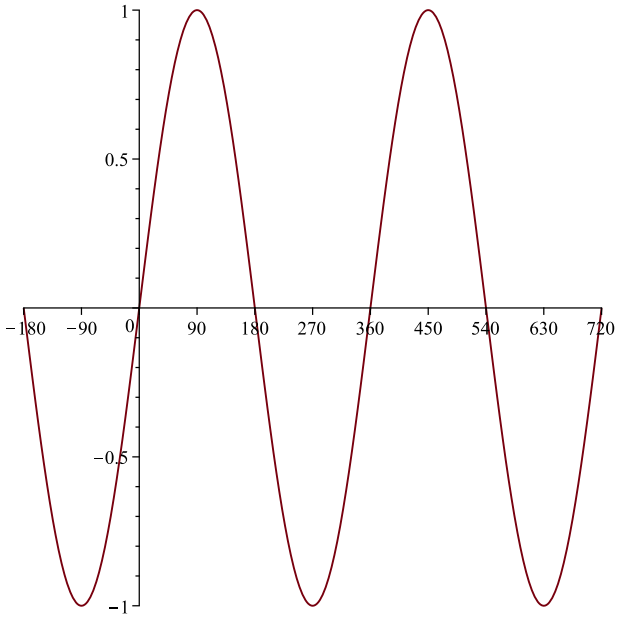


Figure 7.4.5 Graph of Sine

From the first section we know that the graph of sine is a wave that repeats. The piece that is repeated is called a **cycle**. In the default graph this is from 0° to 360° as shown in Figure 7.4.4.

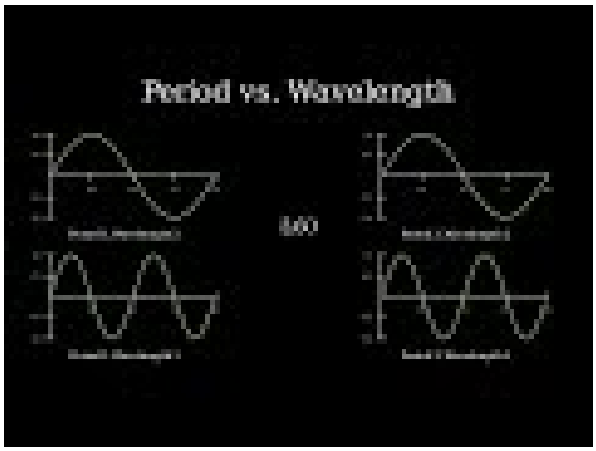
The length of the cycle can be modified. Depending on the application we interpret and measure the length of the cycle differently. This section defines two traits of the cycle, illustrates the traits, and gives an alternate definition for one trait.

Definition 7.4.6 Period. The length of a cycle measured in time is called the **period**. ◆

Definition 7.4.7 Wavelength. The length of a cycle measured in distance is called the **wavelength**. ◆

The video below illustrates the difference between wavelength and period of a sine wave. The top two sine curves have the same wavelength (2), and the bottom two sine curves have the same wavelength (1). We can see this because the top two curves have one cycle from 0 to 2, and the bottom two curves have one cycle from 0 to 1. This implies the bottom curves have a wavelength that is half that of those on top.

The left two sine curves have the same period. They complete one cycle in one (1) seconds. The right two sine curves have the same period. They complete one cycle in two (2) seconds.



Standalone
In Context

Sometimes instead of measuring how long a single cycle is in units of time, we measure how many cycles occur in a fixed unit of time. This is called **frequency**.

Definition 7.4.8 Frequency. The number of cycles that occur per second is called the **frequency**. This is typically measured in Hertz (Hz). 1 Hz is one cycle per second. ◆

Note that frequency is the inverse of the period as shown below. In the video the sine waves above, the sine waves on the right have a period of two (2) and a frequency of 1/2.

Table 7.4.9 Period and Frequency are Inverses

Period	Frequency
$\frac{n \text{ seconds}}{1 \text{ cycle}}$	$\frac{n \text{ cycles}}{1 \text{ second}}$

Example 7.4.10 If a wave has a period of 1/3 seconds, what is its frequency?

We can see how many 1/3 of a second there are in one second. That is

$$\begin{aligned} f \cdot \frac{1}{3} &= 1 \\ f &= 3 \end{aligned}$$

The frequency is 3.

We can consider this a conversion of units. If the period is

$$\frac{1/3 \text{ seconds}}{\text{cycle}}$$

and frequency is in cycles per second what we want is to remove the 1/3 from the denominator (turn it into 1).

$$\begin{aligned} \frac{\text{cycles}}{1/3 \text{ seconds}} &= \\ \frac{\text{cycles}}{1/3 \text{ seconds}} \cdot \frac{3}{3} &= \frac{3 \text{ cycles}}{\text{second}} \end{aligned}$$

▲

Generally, if the period is T then the frequency is

$$f = \frac{1}{T}.$$

Example 7.4.11 What are the period and wavelength of middle C which has a frequency of 261.63 Hz? This is a measurement in a science model so we will use significant digits.

Because we know the frequency we can directly calculate the period.

$$\begin{aligned} 261.63 &= \frac{1}{T}. \\ 261.63 \cdot T &= \frac{1}{T} \cdot T. \\ 261.63T &= 1. \\ \frac{T \cdot 261.63}{261.33} &= \frac{1}{261.33}. \\ T &= \frac{1}{261.63} \\ &\approx 0.003822\underline{1}916 \\ &\approx 0.0038222. \end{aligned}$$

For the wavelength we need to know that the speed of sound is 1116 feet/second. Now we can use the fact that Hz is cycles per second to convert frequency (cycles per second) to wavelength (feet per cycle). The units suggest that we multiply as follows.

$$\frac{\text{second}}{261.63 \text{ cycles}} \cdot \frac{1116 \text{ feet}}{\text{second}} \approx 4.265\underline{5}65875 \approx 4.266 \frac{\text{feet}}{\text{cycle}}.$$

▲

Example 7.4.12 A local AM radio station broadcasts at 750.0 Hz. Note radio waves move at the speed of light which is approximately 2.9979×10^8 meters per second. What are the period and wavelength of this radio signal?

Solution. Because we know the frequency we can directly calculate the period.

$$\begin{aligned} T &= \frac{1}{750.0} \\ &\approx 0.001333333333 \\ &\approx 0.00133 \end{aligned}$$

For the wavelength we need to convert units from seconds per cycle to meters per cycle

$$\frac{1 \text{ s}}{750.0 \text{ cycles}} \cdot \frac{2.9979 \times 10^8 \text{ m}}{\text{s}} \approx 399720 \frac{\text{feet}}{\text{cycle}} \approx 399700 \frac{\text{m}}{\text{cycle}}.$$

▲

Checkpoint 7.4.13 Consider the note A3 which has frequency 220 Hz.

What is the period for A3? _____

What is the wavelength for A3? _____ ft

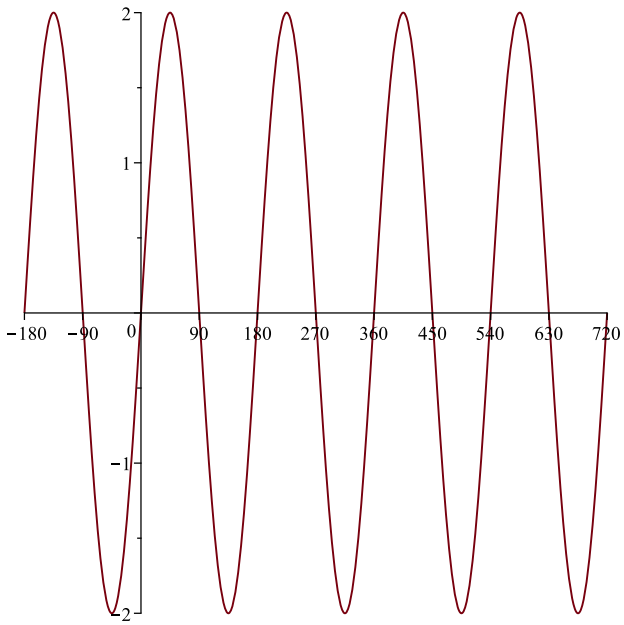
Recall that the speed of sound is 1116 feet per second.

These are scientific measurements; round accordingly.

Frequency, period, and wavelength are all about how fast a sine wave moves. Amplitude is about how strong it is.

Definition 7.4.14 Amplitude. The height of the wave (from center to top) is called the **amplitude**. ◆

Example 7.4.15



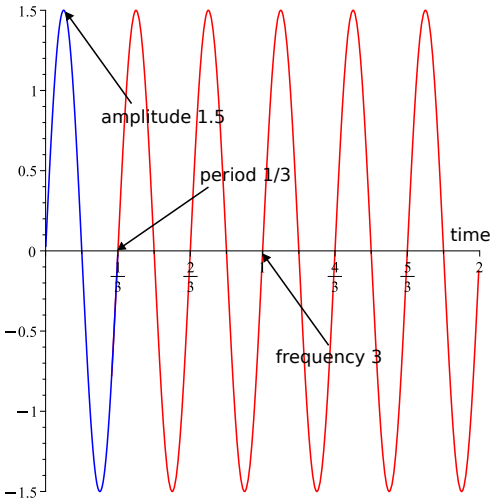
The amplitude of this sine wave is 2. The period is 180. Note without a context period and wavelength are the same. The frequency is

$$f = \frac{1}{180} \approx 0.0056$$

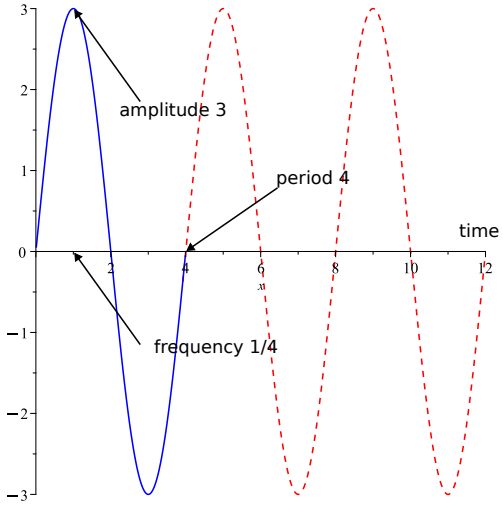
▲

In applications the wavelength, period/frequency, and amplitude are determined experimentally: we

rarely see the wave. However, in order to practice distinguishing between these properties, we will use visual examples and exercises.



(a) Sine Wave



(b) Another Sine Wave

Figure 7.4.16 Identifying Sine Wave Properties

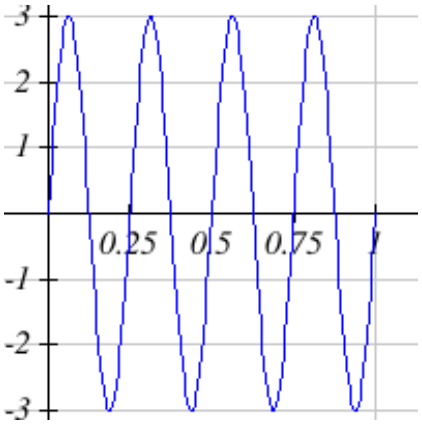
Example 7.4.17 Consider the sine wave in Figure 7.4.16(a). The amplitude is 1.5 because the wave reaches a max of 1.5 and a min of -1.5.

The period is 1/3, because a full cycle ends at 1/3. The frequency is 3, because there are three copies of the cycle from 0 to 1.

Switch to the sine wave in Figure 7.4.16(b). The amplitude is 3 because the wave reaches a max of 3 and a min of -3.

The period is 4, because a full cycle ends at 4. The frequency is 1/4, because the cycle is only 1/4 complete by 1. ▲

Checkpoint 7.4.18



Note the graph shows one second.
What is the frequency of this wave? ____
What is the period of this wave? ____
What is the amplitude of this wave? ____

7.4.3 Transformations of Sine

In Subsection 5.4.1 and Subsection 6.2.2 we learned how to transform a graph by shifting it and reflecting it. Those apply to trigonometric graphs as well. This section illustrates vertical and horizontal stretches and connects them to amplitude and wavelength.

Activity 7.4.19 This activity demonstrates how the sine function is modified to change amplitude. Use Figure 7.4.21 to answer the following. Note the amplitude of the unmodified graph is 1.

(a) If you set $a = 2$, that is graph $2\sin(\pi x)$ what is the amplitude?

(b) If you set $a = 3$, that is graph $3\sin(\pi x)$ what is the amplitude?

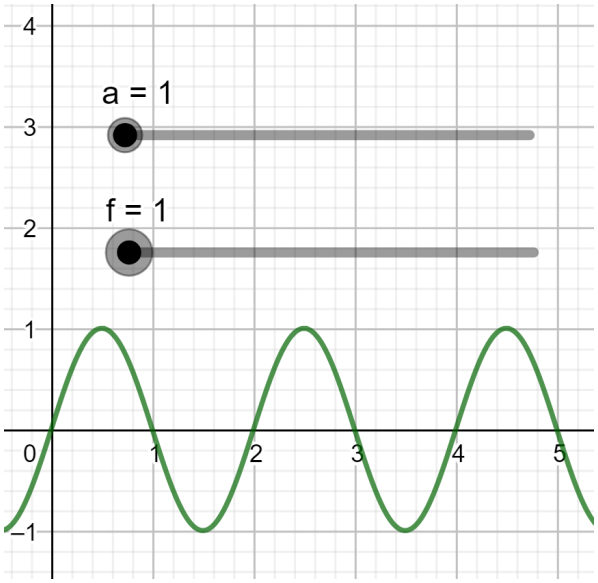
(c) How could you obtain an amplitude of $1/2$?

Activity 7.4.20 This activity demonstrates how the sine function is modified to change wavelength. Use Figure 7.4.21 to answer the following. Note the wavelength of the unmodified graph is 2.

(a) If you set $f = 2$, that is graph $\sin(2\pi x)$ what is the wavelength?

(b) If you set $f = 3$, that is graph $\sin(3\pi x)$ what is the wavelength?

(c) How could you obtain a wavelength of 4?



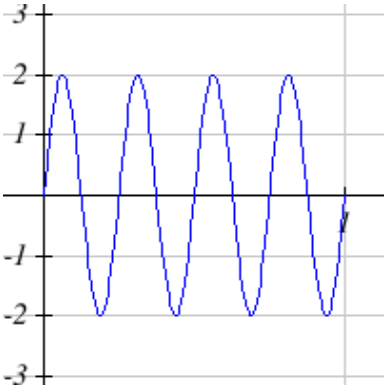
Standalone
In Context
Embed

Figure 7.4.21 Amplitude and Wavelength

7.4.4 Exercises

Frequency and Amplitude. Use the definitions of these properties to answer the questions.

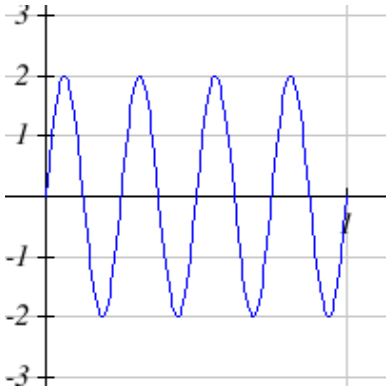
1.



What is the frequency of this wave? _____
Note the graph shows one second.

What is the amplitude of this wave? _____

2.

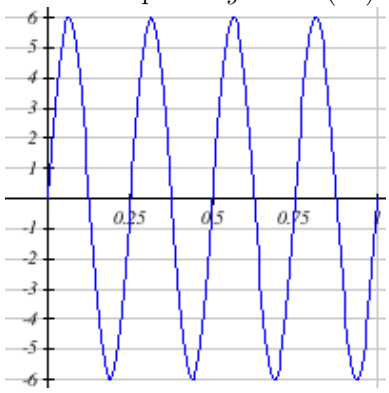


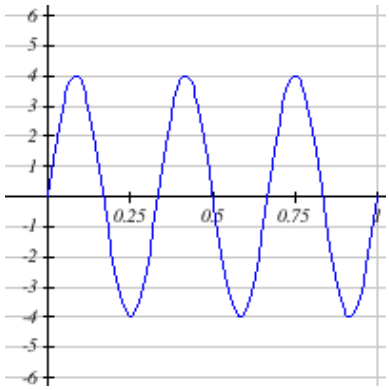
What is the frequency of this wave? _____
Note the graph shows one second.
What is the amplitude of this wave? _____

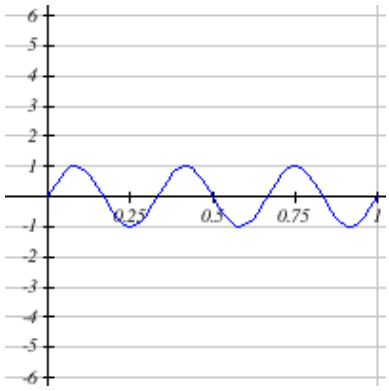
3. Find the period (in seconds) of a wave whose frequency is 7 KHz. _____
Write to two, non-zero decimal places.

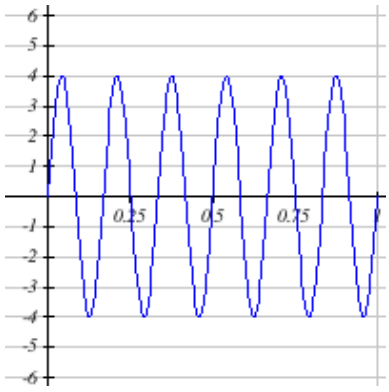
4. What is the frequency in Hz of a wave with period 0.25? _____
Write accurate to 2 decimal places.

5. Which of the following graphs is the correct plot of $y = 4\sin(6x)$?

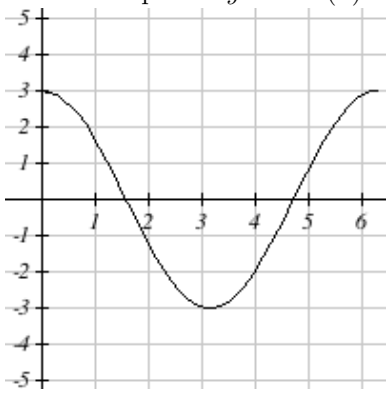


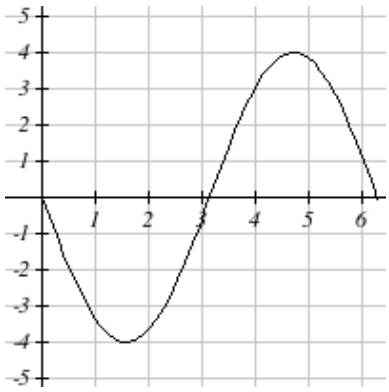


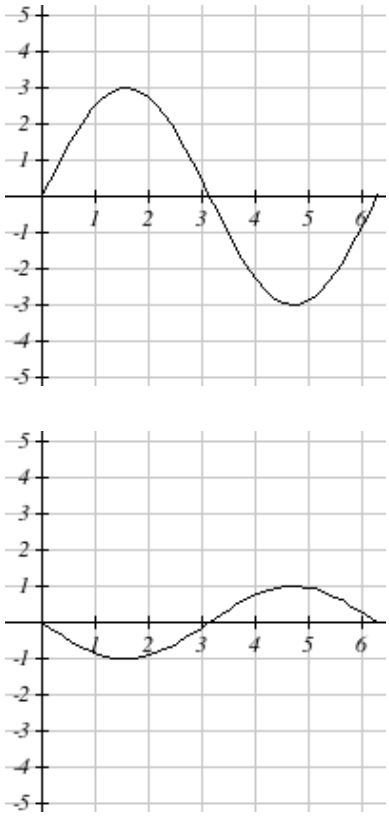




6. Which of the following graphs is the correct plot of $y = 3\sin(x)$?







Applications. Apply your knowledge of frequency and amplitude to answer these wave questions.

7. A local AM radio station broadcasts at 1710 Hz. Radio waves travel at the speed of light which is approximately 3.0×10^8 meters/second. What is the wavelength? _____
Round to the units (1's).

8. Recall that light is a wave and so we can use the relationship that the speed of light is the wavelength of a photon multiplied by the frequency of a photon. $c = \lambda f$. A famous photon in astronomy is the photon emitted by a hydrogen atom with a frequency of $f = 1440$ MHz. What is the wavelength of this photon in centimeters? _____ centimeters

9. Near ultraviolet light has a wavelength of 300 nanometers. Note a nanometer is 1×10^{-9} of a meter. The speed of light is approximately 3×10^8 meters/second. What is the frequency in Hertz?
Enter your answer in scientific notation with three significant figures. For example: 3.29E13 or 4.29E-12.
_____ Hz

10. The frequency of a vibrating object triples. By what factor does the period change? *You can enter a decimal or fraction.*
The period is _____ times what it had been.

7.5 Project: Effects of Scale on Error

Project 7.5.1 Select the Best Time of Day. In Example 7.1.17 we learned to indirectly measure height of an object from the length of its shadow and angle of elevation. Here we consider the effects of the angle of elevation on the precision of our result.

(a) Suppose the shadow of a tree is measured to be 73 ft. If the the angle of elevation of the sun from the

end of the shadow is 45°, what is the height of the tree.

(b) We will consider the impact of the sun's angle of elevation on the calculation. For this section suppose the tree is exactly 73 ft tall.

- **(i)** Suppose the angle of elevation of the sun is 45°. What is the length of the shadow?
- **(ii)** Suppose the angle of elevation of the sun is 55°. What is the length of the shadow?
- **(iii)** Suppose the angle of elevation of the sun is 65°. What is the length of the shadow?
- **(iv)** Suppose the angle of elevation of the sun is 75°. What is the length of the shadow?
- **(v)** Suppose the angle of elevation of the sun is 85°. What is the length of the shadow?
- **(vi)** As the angle of elevation grows from 0° toward 90° does the length of the shadow increase directly or inversely? Is it linear?
- **(vii)** Note the angle of elevation of the sun grows from morning until (high) noon and then decreases again. At what time of day would it be easiest to measure the length of the shadow in order to estimate the height of the tree?

(c) We will consider the effect of error in measurement of the angle of elevation of the sun on our calculation of the height of the tree.

- **(i)** Suppose the shadow's length is 73 ft. What is the estimated height of the tree if the angle of elevation is measured to be 46°? 44°?
- **(ii)** Suppose the shadow's length is 51 ft. What is the estimated height of the tree if the angle of elevation is measured to be 56°? 54°?
- **(iii)** Suppose the shadow's length is 34 ft. What is the estimated height of the tree if the angle of elevation is measured to be 66°? 64°?
- **(iv)** Suppose the shadow's length is 20 ft. What is the estimated height of the tree if the angle of elevation is measured to be 76°? 74°?
- **(v)** Suppose the shadow's length is 6 ft. What is the estimated height of the tree if the angle of elevation is measured to be 86°? 84°?
- **(vi)** How much effect on the estimated height of the tree can error in measurement of the angle have?
- **(vii)** Does the error change as the angle of elevation increases?

(d) We will consider the effect of error in measurement of the length of the shadow on our calculation of the height of the tree.

- **(i)** Suppose the angle of elevation of the sun from the end of the shadow is 45°. What is the estimated height of the tree if the length of the shadow is measured to be 72 ft? 74 ft?
- **(ii)** Suppose the angle of elevation of the sun from the end of the shadow is 55°. What is the estimated height of the tree if the length of the shadow is measured to be 50 ft? 52 ft?
- **(iii)** Suppose the angle of elevation of the sun from the end of the shadow is 65°. What is the estimated height of the tree if the length of the shadow is measured to be 33 ft? 35 ft?
- **(iv)** Suppose the angle of elevation of the sun from the end of the shadow is 75°. What is the estimated height of the tree if the length of the shadow is measured to be 19 ft? 21 ft?
- **(v)** Suppose the angle of elevation of the sun from the end of the shadow is 85°. What is the estimated height of the tree if the length of the shadow is measured to be 5 ft? 7 ft?
- **(vi)** At what angle of elevation does the difference in shadow length make the greatest difference?
- **(vii)** What does this suggest about when we should measure the shadow?

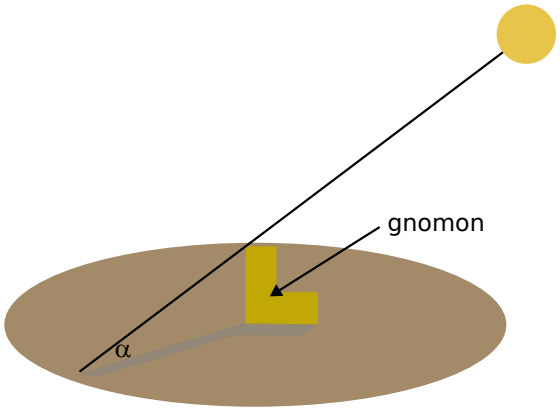
7.6 Project: A Clever Sundial

Project 7.6.1 How Tall is the Gnomon? Sundials are one of the oldest types of clocks. Because the apparent motion of the sun through the sky defines time for us, the position of a shadow is an obvious way to measure that progress. One common type of sundial has a vertical object, called a **gnomon**, standing in the middle of a circle. If the circle is oriented correctly one edge of the gnomon's shadow cross the number marking the time.

We also notice that the sun rises and sets at different times throughout the year and that its path across the sky changes. In particular the sun is much higher in the sky in summer and lower in winter. This change is especially noticable in the far north and far south of the planet.

In this project we calculate the effect of the shifting path on shadow length and consider ways to adjust for that. This is accomplished using simple trigonometric calculations.

(a) First, we will consider how long the gnomon's shadow is each month of the year. For convenience we will use The first of each month, and "high" noon (when the sun is highest in the sky). The location for the following data is in Anchorage, Alaska.



(i) **Figure 7.6.2** Sundial with altitude angle labeled

For each month calculate the expected length of the gnomon's shadow. The gnomon is 22.0 cm tall, and the table below gives the altitude for the first of each month at high noon (azimuth 180°).

Table 7.6.3 Altitude and Shadow Length for Sundial

Date	Altitude	Shadow Length
January	6.04°	
February	12.07°	
March	21.64°	
April	33.76°	
May	44.21°	
June	51.02°	
July	51.88°	
August	46.61°	
September	36.77°	
October	25.30°	
November	14.16°	
December	7.00°	

(b) Here we consider an idea for making this sundial fit in a fixed space.

(i) Based on the calculations in the table above can this sundial fit into a circle of one meter diameter?

(ii) If we want to make all the shadows fit on a sundial (circle) of one meter diameter, how tall should the gnomon be?

(iii) Alternately we could construct a gnomon that was raised and lowered so that the length of the shadow at high noon remains the same every day. Suppose we want the shadow to be 48 cm long at noon. Calculate the necessary gnomon height for each entry in the table below.

Table 7.6.4 Calculated Gnomon Height for Fixed Shadow Length

Date	Altitude	Gnomon Height
January	6.04°	
February	12.07°	
March	21.64°	
April	33.76°	
May	44.21°	
June	51.02°	
July	51.88°	
August	46.61°	
September	36.77°	
October	25.30°	
November	14.16°	
December	7.00°	

(iv) Suppose that we construct gnomon such that it rises out of the middle of the sundial so the height above the face of the sundial is the calculated height. If we need at least 10 cm of the gnomon below the top of the sundial, how tall must the gnomon be based on your table above?

7.7 Project: Wind Correction Angle

Project 7.7.1 Wind Correction Angle. In flight the direction and speed across the ground of aircraft depends on the speed and direction it is facing and the speed and direction of the wind. In this project we will learn how to combine the two to determine the actual direction and speed and then figure out how to select a direction for the aircraft in order to follow an intended ground track.

(a) These steps walk through converting the angle and speed into x and y components which may be added. The result is an x and y component for the ground track that we can convert back into an angle and speed. Pilots perform this calculation using a device. We do it manually here to see how those devices perform the calculation.

For this task, the aircraft has a heading of 115° and speed of 95 nm/hr. The wind has a heading of 55° and speed of 12 nm/hr. Because we measure angles from straight up instead of to the right, x is the vertical component here and y is the horizontal. This also begins with headings (straight up is north or 0°) instead of angles. If needed, you can review changing headings to angles in Example 7.3.14

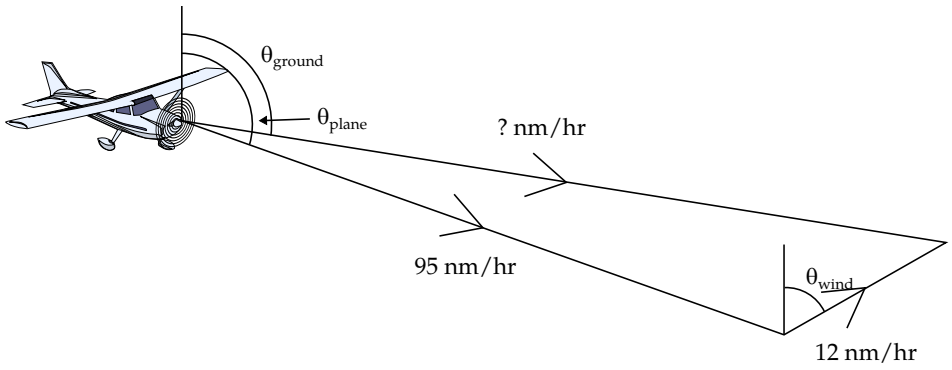


Figure 7.7.2 Wind Correction Triangle

(i) To calculate the x and y components of the plane, we use the right triangle with the 95 nm/hr side as the hypotenuse. The angle in this triangle is the supplement of θ_{plane} that is $\alpha_{plane} = 180° - \theta_{plane}$. Calculate this angle.

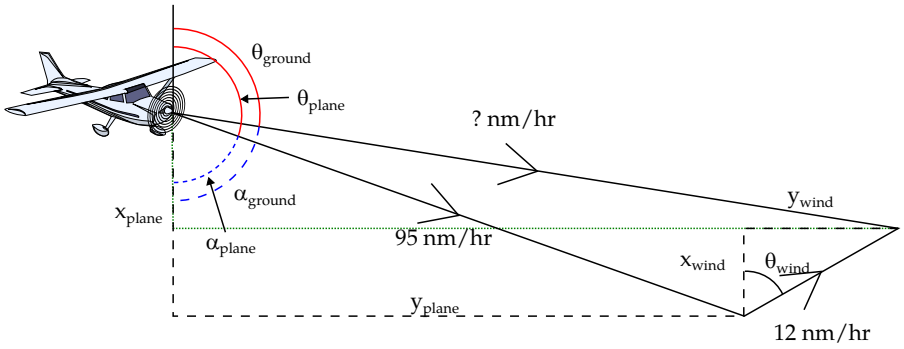


Figure 7.7.3 Wind Correction Triangle

(ii) Calculate x_{plane}.

(iii) Calculate y_{plane}.

(iv) To calculate the x and y components of the wind, we use the right triangle with the 12 nm/hr side as the hypotenuse (small one on the bottom right of the diagram). The angle in this triangle is $\theta_{wind} = 55°$.

Calculate the x and y components.

(v) To calculate the x component of the ground track we use the right triangle with the unknown nm/hr side as the hypotenuse. In the diagram it has green, dotted legs. What is the x component of the ground track?

(vi) What is the y component of the ground track?

(vii) Now that we know the x and y components of the ground track (legs of the right triangle), calculate the ground speed. This is the length of the hypotenuse.

(viii) Using the side lengths you just calculated, calculate the angle α_{ground}.

(ix) Calculate the ground heading (i.e., θ_{ground}).

(b) When flying the needed ground track (where we are going) and wind are given as is the aircraft speed. We need to calculate the heading for the aircraft to maintain that track. This somewhat reverses the process above. Follow these steps to calculate the required heading (θ_{plane}) when we want a ground track (heading) of $\theta_{ground} = 100°$. As before our wind is 12 nm/hr from $\theta_{wind} = 55°$.

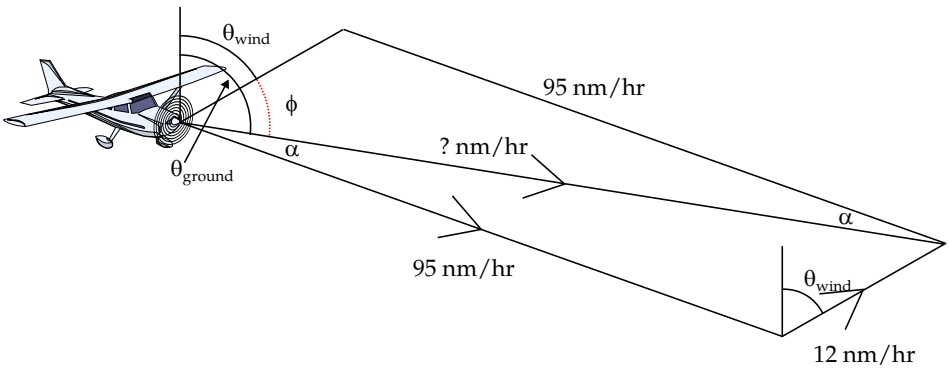


Figure 7.7.4 Wind Correction Triangle

For this calculation we will use this alternate version of the wind correction diagram. Specifically we will use the top triangle. Because it consists of the same three side lengths and directions, the two triangles are similar (actually congruent). It is easier to calculate some angles on this version of the triangle.

(i) Calculate ϕ, the angle in the triangle.

(ii) We know ϕ and the side length opposite it, and we want to know α (wind correction angle), and we know the side length opposite it. Set up Law of Sines for these angle/side pairs, then solve for α.

(iii) α is the wind correction angle (how much to turn from the desired ground track). What is the heading (angle from vertical in the diagram) the plane will be facing?

(iv) Calculate the measure of the third (unlabeled) angle of the triangle.

(v) Calculate the length of the third side of the triangle. This is the ground speed.

(c) Finally we consider a quantitative literacy question about the effect of a cross wind on efficiency. When the wind is fully a tailwind (blowing from behind in the same direction we are flying), then the ground speed is the sum of the airspeed and wind speed. When the wind is fully a head wind (blowing from in front in the opposite direction from which we are flying), then the ground speed is the difference of the airspeed and wind speed. The first question then is what is the effect of a wind that is perpendicular to our ground track (neither tail nor head).

(i) Use the method from above to calculate wind correction angle and ground speed if the ground track is 90°, the wind is from 180° at 12 knots, and the airspeed is 95 knots.

(ii) Use the method from above to calculate wind correction angle and ground speed if the ground track is 90°, the wind is from 180° at 24 knots, and the airspeed is 95 knots.

(iii) Does a pure crosswind (no head or tail component) slow down or speed up cruise flight?

(iv) The final question is whether a wind with a tailwind component always increases ground speed. Calculate the wind correction angle and ground speed if the ground track is 90°, the wind is from 225° at 12 knots, and the airspeed is 95 knots.